Unit - I

Lesson - 1

LEGENDRE POLYNOMIALS

Objective of the lesson :

* To find a solution of Legendre differential equation in series.
* To project the generating function in a series of Legendre polynomials
* To derive some of the recurrence relations.
* To find the differential and integral representations of the Legendre polynomials.
* To express the derivative of the Legendre Polynomial into a series of Legendre Polynomials.
* To obtain the associated Legendre Polynomials.

Structure of the lesson :

1.1 Introduction

All the lessons in this unit deal with the solutions of second order differential equations with variable coefficients. The power series method yields solutions of differential equations in the form of power series. The solutions 'y' of a given differential equation is assumed in the form of a power series with undetermined coefficients and the coefficients are determined successively by inserting that series and the series ofr the derivatives of 'y' into the given equation.

The practical use of this method has in computing values of the solutions and deriving general properties of the solution and deriving general properties of the solution.

1.2 Basics of Power Series Solutions :

When the function $f(x)$ is expressed in the power series

$$f(x) = \sum_{m=0}^{\infty} a_m (x-\alpha)^m \quad \text{---------- (1)}$$

then the series converges for all x in the interior of the interval, that is, for all x for which $|x-\alpha| > R$ and diverges when $|x-\alpha|<R$ and diverges when $|x-\alpha|>R$. The interval may also be infinite in which case the series is said to converge for all x.

The quantity R is called the radius of convergence of (1); it is the distance of each end point of the convergence interval from the centre α. If the series converges for all x, then we set R = α.

A function $f(x)$ is said to be analytic at a point $x=\alpha$ if it can be represented by a power series in powers of $(x-\alpha)$ as in (1) with radius of convergence R > 0.

In this connection, an important theorem (without proof) is to be noted.

"If the function f, g and r in the differntial equation

$$y'' + f(x)y' + g(x)y = r(x) \quad \text{---------- (2)}$$

are analytic at $x=\alpha$, then every solution $y(x)$ of (2) is analytic at $x=\alpha$ and can thus be represenented by a power series in powers of $(x-\alpha)$ with radius of convergence R > 0"

In applying this theorem, it is important to write the linear differential equation in the form (2) with 1 as the coefficient of y''.

A point $x=\alpha$ is said to be a singularity of (2) if one or more of the functions, f, g, and r are not analytic at $x=\alpha$.

If $x=\alpha$ is a singularity of (2) and the product functions $(x-\alpha)\ f(x)$ and $(x-\alpha)^2\ g(x)$ both are analytic at $x=\alpha$, then $x=\alpha$ is called as regular singularity. If both or any one of the product functions are not analytic at $x=\alpha$, then that is said to be an irregular singularity.

If $r(x)$ is zero in (2), (i.e.) in the differential equation

$$y''+f(x)y'+g(x)y=0 \text{ -----------(3)}$$

if $x=\alpha$ is an irregular similarity, then it is too difficult to find the series solution of (3). However, if it has a regular singularity at $x=\alpha$, then the series solution of (3) can be found in the neighbourhood of α. In this case, Frobenius introduced a series solution

$$y=(x-\alpha)^k\left[a_0+a_1(x-\alpha)+a_2(x-\alpha)^2+......\right] a_0\neq 0 \text{ --------- (4)}$$

Which is known as a Frobenius series. When k = 0, (4) reduces to the usual Taylor series which will be a special case of Frobenies series.

The process of finding Frobenius series solution will be applied to all the differential equations occuring in this and ensuing lessons.

1.3 Solution of Legendre differential equations :

The Legendre differential equation is given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \text{ ----- (5)}$$

There is no singularity at x = 0 so that we can obtain the solution in the form of an ascending or descending series developed about x = 0.

Let m assume the series solution as

$$y=\sum_{r=0}^{\infty} a_r\, x^{k-r}\ ,\quad a\neq 0 \text{ ---------- (6)}$$

$$\therefore\frac{dy}{dx}=\sum_{r=0}^{\infty}(k-r)a_r\ x^{k-r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}(k-r)(k-r-1)a_r\ x^{k-r-2}$$

Substituting these values in (5), we have

$$\sum_{r=0}^{\infty}\left[\left(1-x^2\right)(k-r)(k-r-1)x^{k-r-2}-2x(k-r)x^{k-r-1}+n(n+1)x^{k-r}\right]a_r=0$$

or

$$\sum_{r=0}^{\infty}\left[(k-r)(k-r-1)x^{k-r-2}+\{n(n+1)-(k-r)(k-r+1)\}x^{k-r}\right]a_r \equiv 0 \text{ ------ (7)}$$

Since equation (7) is an identity and therefore the coefficients of various powers of x can be equated to zero.

Let us first equate the coefficient of x^k, the highest power of x (by putting r = 0 in (7)) to zero, there we get

$a_0\left[n(n+1)-k(k+1)\right]=0$, called the <u>indicial equation</u>. Since $a_0 \neq 0$, thereby $(n-k)(n-k+1)=0$ or

$$k=n \text{ or } -(n+1) \text{ ----------- (8)}$$

Again equating the coefficient of x^{k-1} to zero, by putting r = 1 in (7), we have

$$\left[n(n+1)-(k-1)k\right]a_1=0 \text{ ------- (9)}$$

Since the values of k are fixed by (8), the expression in brackets in (9) cannot vanish and thus it leads to $a_1=0$. Let us now equate the coefficient of x^{k-r} in (7) to zero. We get the recurrence relation as

$$a_r = \frac{(k-r+2)(k-r+1)}{n(n+1)-(k-r)(k-r+1)}a_{r-2} \text{ ------- (10)}$$

Case : I When k = n from (8), we have

$$a_r = -\frac{(n-r+2)(n-r+1)}{r(2n-r+1)}a_{r-2} \text{ ------ (11)}$$

Putting r = 2, 3, ... in (11), we get

$$a_2=-\frac{n(n-1)}{2(zn-1)}a_0$$

$$a_4=-\frac{(n-2)(n-3)}{4(2n-3)}a_2=(-1)^2\frac{n(n-1)(n-2)(n-3)}{2.\,4.\,(2n-1)(2n-3)}$$

$$a_{2r}=(-1)^r\frac{n(n-1)(n-2)......(n-2r+1)}{2.4..........2r\quad(2n-1)(2n-3).....(2n-2r+1)}a_0$$

All the a's having odd suffixes are zero since a_1 vanishes.

So the series solutions when k = n is

$$y=a_0\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}........\right] \text{-------- (12)}$$

where a_0 is an arbitrary constatn. If a_0 is choosen as

$\frac{1.3.5......(2n-1)}{\lfloor n}$. where n is a +ve integer then the solution is designated as $P_n(x)$ which is called Legendre Polynomial.

So,

$$P_n(x)=\frac{1.3.5.....(2n-1)}{\lfloor n}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}.....\right] \text{------- (13)}$$

Case - II : When $k=-(n+1)$, the recurrence relation among the coefficients has the form

$$a_r=\frac{(n+r-1)(n+r)}{r(2n+r+1)}a_{r-2} \text{ ------- (14)}$$

As a_r is already shown to be zero, in this case also, only even suffixes of a's will remain and th esolutions contains a series of positive term as

$$y=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-2}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+......\right] \text{-------- (15)}$$

When the arbitrary constant a_0 is choosen as $\frac{\lfloor n}{1.3.5.....(2n+1)}$ where n is a +ve integer the solution is known as

$$Q_n(x)=\frac{\lfloor n}{1.3.5...(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-2}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}+.....\right] \text{------- (16)}$$

The most general solution of the Legendre's equation is

$$y=AP_n(x)+bQ_n(x) \text{ ----- (17)}$$

where A and B are arbitrary constants.

Note : *For positive integral values for n, the solutions* $P_n(x)$ *and* $Q_n(x)$ *have great utility. Then* $P_n(x)$ *is called Legendre polynomial, while* $Q_n(x)$ *is an infinite series.*

Example (1) : Evaluate the values of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$.

Solution : We know the Legendre polynominal $P_n(x)$ as given by (18) for +ve integral values of n. Putting n = 0, 1, 2, 3, and 4, we get

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= \frac{1}{\lfloor 1} x^1 = x \\ P_2(x) &= \frac{1.3}{\lfloor 2}\left[x^2 - \frac{2.1}{2.3}x^0\right] \\ &= \frac{3x^2-1}{2} \\ P_3(x) &= \frac{1.3.5}{\lfloor 3}\left[x^3 - \frac{3.2}{2.5}x\right] \\ &= \frac{5x^3-3x}{2} \\ \text{and } P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \end{aligned}\right\} \text{----------(18)}$$

(Other terms vanish)

Note : *Remember that* $P_n(x)$ *is a polynomial of order n.*

Example 2 : Express $f(x) = 5x^3 + 6x^2 - 8x + 4$ as a linear combination of Legendre polynomials.

Solution : Since we know that the Lengendre polynomial $P_n(x)$ is order n and since the given function $f(x)$ is a third order polynomial, we can write

$$f(x) = 5x^3 + 6x^2 - 8x + 4 = C_0 P_0 + C_1 P_1 + C_2 P_2 + C_3 P_3$$

$$= C_0(1) + C_1(x) + C_2\left[\frac{3x^2-1}{2}\right] + C_3\left[\frac{5x^3-3x}{2}\right]$$

Since we know P_0, P_1, P_2 and P_3 as given in (18)

Now equating the coefficients of like powers of x on both sides, we have

$$\frac{5C_3}{2} = 5 \quad \text{or} \quad C_3 = 2$$

$$\frac{3C_2}{2} = 6 \text{ or } C_2 = 4$$

$$-\frac{3C_3}{2} + C_1 = -8 \therefore C_1 = -5$$

$$-\frac{C_2}{2} + C_0 = 4 \therefore C_0 = 6$$

So the given $f(x)$, can be written as

$$f(x) = 2P_3 + 4P_2 - 5P_1 + 6P_0$$

1.4 Generating function for $P_n(x)$:

Q : Show that $P_n(x)$ is the coefficient of h^n in the expansion of $\left(1-2xh+h^2\right)^{-\frac{1}{2}}$.

OR

Show that $\left(1-2xh+h^2\right)^{-\frac{1}{2}}$ is the generating function of the Legendre Polynomial $P_n(x)$.

Solution : We have

$$\left(1-2xh-h^2\right)^{-1/2} = \left[1-h(2x-h)\right]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \frac{1.3.5.}{2.4.6.}h^3(2x-h)^3 + ... + \frac{1.3.5....(2n-1)}{2.4.6......2n}h^n(2x-h)^n + ...$$

(by Brinomial expansion)

The coefficient of h^n in this expansion

$$= \frac{1.3.5.....(2n-1)}{2.4.6.....2n}\left(2x^n\right) - \frac{1.3.5....(2n-3)}{2.4.6.....(2n-2)}(2x)^{n-2}(n-1)_{C_1} + \frac{1.3.5....(2n-5)}{2.4.6.....(2n-4)}(2x)^{n-4}.(n-2)_{C_2} ......$$

(the terms are obtained by collecting the coefficients from the term containing $h^n(2x-h)^n$, $h^{n-1}(2x-h)^{n-1}, etc....$ in the expansion).

$$=\frac{1.3.5.....(2n-1)}{\lfloor n}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}....\right]=P_n(x)$$

or $$\sum_{n=0} h^n P_n(x)=\left(1-2xh+h^2\right)^{-\frac{1}{2}} \text{ ------ (19)}$$

Corollary : To show that $P_n(1)=1$

We know that

$$\sum_{n=0} h^n P_n(x)=\left[1-2xh+h^2\right]^{-\frac{1}{2}}$$

Put x = 1, $$\sum_{n=0} h^n P_n(1)=\left[1-2h+h^2\right]^{-\frac{1}{2}}=(1-h)^{-1}$$

$$=1+h+h^2+......$$

Equating the coefficient of h^n on both sides, we get

$$P_n(1)=1 \text{ for all n.}$$

1.5 Recurrence relation :

(1) : Show that

$$nP_n(x)=(2n-1)x\,P_{n-1}(x)-(n-1)P_{n-2}(x)$$

Proof : We know that

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}}=\sum_{n=0} h^n P_n(x) \text{ --------- (19)}$$

Differentiating w.r.t. h, we have

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-\frac{3}{2}}2.(h-x)=\sum_{n=0} n h^{n-1}P_n(x)$$

or

$$(x-h)\sum_{n=0} h^n P_n(x)=\left(1-2xh+h^2\right)\sum_{n=0} n h^{n-1}P_n(x)$$

Equating the coefficient of h^{n-1} on both sides,

$$x P_{n-1}(x) - P_{n-2}(x) = n P_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x)$$

(or) $$n P_n(x) = (2n-1)x\, P_{n-1}(x) - (n-1)\, P_{n-2}(x) \quad \text{------------ (20)}$$

(2) : Show that $P'_{n+1}(x) + P'_{n+1}(x) + P'_{n-1}(x) = 2x P'_n + P_n(x)$

Proof : We know that

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0} h^n P_n(x) \quad \text{---------- (19)}$$

Differentiating (19) w.r.t. to x, we get

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-\frac{3}{2}}(-2h) = \sum_{n=0} h^n P'_n(x)$$

or $$h \sum_{n=0} h^n P_n(x) = \left(1-2xh+h^2\right) \sum_{n=0} h^n P'_n(x)$$

Equating the coefficient of h^n on both sides,

$$P_{n-1}(x) = P'_n(x) - 2x P'_{n-1}(x) + P'_{n-2}(x)$$

or $$P_n(x) = P'_{n+1}(x) - 2x P'_n(x) + P'_{n-1}(x) \quad \text{---------- (21)}$$

(3) : Show that $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Proof : We know the recurrence relation

$$n\, P_n(x) = (2n-1)x\, P_{n-1}(x) - (n-1)P_{n-2}(x) \quad \text{-------- (20)}$$

and $$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$$

or $$P_{n-1}(x) = P'_n(x) - 2x P'_{n-1}(x) + P'_{n-2}(x) \quad \text{-------- (21)}$$

Differentiating (20), we have

$$n P'_n(x) = (2n-1)P_{n-1}(x) + (2n-1)x P'_{n-1}(x) - (n-1)P'_{n-2(x)} \quad \text{------ (22)}$$

Formation of $\left[(2m-1)(21)+2(22)\right]$ gives

$$2n P'_n(x) + (2n-1)P_{n-1}(x) = 2(2n-1)P_{n-1}(x) + (2n-1)P'_n(x) - 2(n-1)P'_{n-2}(x) + (2n-1)P'_{n-2}(x)$$

or $$(2n-1)P_{n-1}(x) = P'_n(x) - P'_{n-2}(x)$$

or $$(2n+1)\, P_n(x) - P'_{n+1}(x) - P'_{n-1}(x) \quad \text{--------- (23)}$$

(4) : Show that $n P_n = x P_n' - P_{n-1}'$

Proof : We know that

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n\left(x\right) \text{ --------- (19)}$$

Differentiating w.r.t. to h, we get

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-\frac{3}{2}}\left(-2x+2h\right) = \sum_{n=0} n h^{n-1} P_n\left(x\right)$$

or $$\left(x-h\right)\left(1-2xh+h^2\right)^{-\frac{3}{2}} = \sum n\, h^{n-1} P_n\left(x\right) \text{ ---------- (24)}$$

Again differentiating (19) w.r.t. to x, we have

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-\frac{3}{2}}\left(-2h\right) = \sum_{n=0} h^n P_n'\left(x\right)$$

or $$h\left(1-2xh+h^2\right)^{-\frac{3}{2}} = \sum_{n=0} h^n P_n'\left(x\right) \text{ ---------- (25)}$$

Now on dividing (24) by (25), we obtain

$$\frac{x-h}{h} = \frac{\sum n\, h^{n-1} P_n\left(x\right)}{\sum h^n\, P_n'\left(x\right)}$$

or $$\left(x-h\right)\sum h^n P_n'\left(x\right) = h\sum n h^{n-1} P_n\left(x\right)$$

on equating the coefficients of h^n on both sides

we get $$n P_n\left(x\right) = x P_n'\left(x\right) - P_{n-1}'\left(x\right) \text{ ---------------- (26)}$$

(5) : Show that

$$\left(n+1\right) P_n\left(x\right) = P_{n+1}' - x\, P_n' \text{ ------------------ (27)}$$

Proof : By adding the recurrence relation (21) and (26), we get the above relation

1.6 Rodrigue's formula : Differential representation of $P_n(x)$

Show that $P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n (x^2-1)^n}{dx^n}$

Proof : Let $y=(x^2-1)^n$ ------------- (28)

$$\therefore \frac{dy}{dx}=n(x^2-1)^{n-1} 2x$$

or $(x^2-1)\frac{dy}{dx} = n(x^2-1)^n 2x = 2n\,xy$ ------- (29)

Differentiating (29) (n+1) times by Leibnitz's theorem, we get

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}C_1 \frac{d^{n+1}y}{dx^{n+1}}(2x) + {}^{n+1}C_2 \frac{d^n y}{dx^n}(2) = 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_1 \frac{d^n y}{dx^n}.1\right]$$

(i.e.,) $(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2(n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^n y}{dx^n}$

or $(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+2}} + 2x(n+1-n)\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^n y}{dx^n} = 0$

(i.e.,) $(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^n y}{dx^n} = 0$

or $(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0$ ----------- (30)

Putting $\frac{d^n y}{dx^n} = V$ in (30) gives

$(1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dv}{dx} + n(n+1)V=0$ ------------- (31)

Which shows that V is a solution of Legendre equation.

Hence $P_n(x)=cV = c\frac{d^n y}{dx^n}$ where c is a constant. ------- (32)

To find c, put x = 1 in both sides of (32)

(i.e.,) $c\left(\dfrac{d^n y}{dx^n}\right)_{x=1} = P_n(1) = 1$ ------- (33)

Again $y = \left(x^2-1\right)^n = (x-1)^n (x+1)^n$

Differentiating both sides n times by Leibnitz's theorem, we get

$$\frac{d^n y}{dx^n} = (x-1)^n \frac{d^n (x+1)^n}{dx^{n-1}} + n\frac{d^{n-1}(x+1)^n}{dx^{n-1}}\left\{n(x-1)^{n-1}\right\} + \ldots\ldots + (x+1)^n \frac{d^n (x-1)^n}{dx^n}$$

Now putting x =1 in the above equation, all the terms in RHS except the last vanish since each term contains the factor (x - 1).

Also $\dfrac{d^n (x-1)^n}{dx^n} = \lfloor n$ $\quad\therefore \left(\dfrac{d^n y}{dx^n}\right) = 2^n \lfloor n$ .... (34)

Hence from (33) and (34), we get

$$2^n (\lfloor n) c = 1 \text{ or } c = \frac{1}{2^n \lfloor n}$$

Now from (32), we get

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n y}{dx^n} = \frac{1}{2^n \lfloor n} \frac{d^n \left(x^2-1\right)^n}{dx^n}$$

Note : *One can find values of* $P_0(x), P_1(x), P_2(x), \ldots\ldots$ *etc., from Rodrigue's formula*

1.7 Orthonormal Property of Legendre Polynominals :

To prove that

(a) $\int_{-1}^{1} P_m(x)\, P_n(x) dx = 0$ if $n \neq m$ Orthogonal property

(b) $\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \dfrac{2}{2n+1}$ if n = m Normalization property.

Proof : (a) Legerdre's differential equation can be written as

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dy}{dx}\right\}+n(n+1)y=0 \text{ ----------- (35)}$$

Since $P_n(x)$ and $P_m(x)$ are solutions of (35), we have

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0 \text{ --------- (36)}$$

and $$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}+m(m+1)P_m=0 \text{ ------- (37)}$$

Simplifying $\left[P_m X(36)-P_n X(37)\right]$, we get

$$P_m\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}-P_n\frac{d}{dx}\left\{\left(1-x^2\right)\frac{d\,P_m}{dx}\right\}+P_n\,P_m\left[n(n+1)-m(m+1)\right]=0$$

Integrating above within given limits, we get

$$\int_{-1}^{1}\left[P_m\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}\right]dx-\int_{-1}^{1}\left[P_n\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}\right]dx\ +(n-m)(n+m+1)\int_{-1}^{1}P_n\ P_m\,dx=0$$

Integration by parts gives us

$$\left[P_m\left(1-x^2\right)P_n'\right]_{-1}^{1}-\int_{-1}^{1}P_m'\left(1-x^2\right)P_n'\,dx-\left[P_n\left(1-x^2\right)P_m'\right]_{-1}^{1}+\int_{-1}^{1}P_n'\left(1-x^2\right)P_m'\ dx$$

$$+(n-m)(n+m+1)\int_{-1}^{1}P_n\,P_m\,dx=0$$

(i.e.,) $(n-m)(n+m+1)\int_{-1}^{1}P_m\,P_n\,dx=0$

or $\int_{-1}^{1}P_m(x)\,P_n(x)dx=0$ if $m\neq n$ ----------------- (38)

(b) : We know that $\left(1-2xh+h^2\right)^{-\frac{1}{2}}=\sum_{n=0}h^n\,P_n(x)$

Squarring both sides, we get

$$\left(1-2xh+h^2\right)^{-1}=\sum_{n=0}^{\infty}h^{2n}\left[P_n(x)\right]^2+2\sum_{n=0}^{\infty}h^{m+n}\,P_m(x)P_n(x)$$

Integrating w.r.t. x between the limits - 1 to +1,

we have $\int_{-1}^{1} \frac{dx}{1-2xh+h^2} = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \left[P_n(x)\right]^2 dx + 2\sum_{n=0}^{\infty} h^{m+n} \int_{-1}^{1} P_m(x) P_n(x) dx$

or $\quad -\frac{1}{2h}\left[\log\left(1-2xh+h^2\right)\right]_{-1}^{1} = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \left[P_n(x)\right]^2 dx$ from (38)

(i.e.,) $\quad -\frac{1}{2h}\left[\log\left(1-h^2\right) - \log(1+h)^2\right] = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \left[P_n(x)\right]^2 dx$

or $\quad \frac{1}{h}\left[\log(1+h) - \log(1-h)\right] = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \left[P_n(x)\right]^2 dx$

(i.e.) $\quad 2\left[1+\frac{h^2}{3}+\frac{h^4}{5}+.........+\frac{h^{2n}}{2n+1}+......\right] = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \left[P_n(x)\right]^2 dx$

Equating the coefficient of h^{2n} on both sides, we get

$$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1} \quad \text{---------- (39)}$$

Both (38) and (39) can be combinedly written as

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\,\delta_{mn}$$

1.8 Integral representation of Legendre Polynomial :

(i) Laplace's first integral for $P_n(x)$

Show that $P_n(x) = \frac{1}{\pi}\int_{0}^{\pi}\left[x \pm \sqrt{x^2-1}\cos\phi\right]^n d\phi$ n being a +ve integer -------- (40)

Proof : We know the standard integral

$$\int_{0}^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad \text{if } a > b \quad \text{--------- (41)}$$

on putting $a = 1 - xh$ and $b = h\sqrt{x^2-1}$ is (41), we get

$$\int_0^{\pi} \frac{d\phi}{1-xh \pm h\sqrt{x^2-1}\,\cos\phi} = \pi\left(1-2xh+h^2\right)^{-\frac{1}{2}}$$

or $$\pi\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \int_0^{\pi}\left[1-h\left\{x\pm\sqrt{x^2-1}\,\cos\phi\right\}\right]^{-1} d\phi \quad \text{-------- (42)}$$

If h is small so that $\left|h\left\{x\pm\sqrt{x^2-1}\,\cos\phi\right\}\right|<1$, then

$$\left[1-h\left\{x\pm\sqrt{x^2-1}\cos\phi\right\}\right]^{-1} = 1+t+t^2+.....= \sum_{n=0} t^n$$

where $t=h\left\{x\pm\sqrt{x^2-1}\,\cos\phi\right\}$

$\therefore (42)$ becomes

$$\pi \sum_{n=0} h^n P_n(x) = \sum_{n=0} h^n \int_0^{\pi}\left\{x\pm\sqrt{x^2-1}\,\cos\phi\right\}^n d\phi$$

Equating the coefficient of h^n on both sides

$$\pi P_n(x) = \int_0^{\pi}\left\{x\pm\sqrt{x^2-1}\,\cos\phi\right\}^n d\phi \quad \text{----------- (40)}$$

(ii) Laplace's second integral for $P_n(x)$:

Show that $$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{x^2-1}\cos\phi\right]^{n+1}} \quad \text{------------- (43)}$$

Proof : We know the standard integral

$$\int_0^{\pi}\frac{d\phi}{a\pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad \text{if } a>b \quad \text{------------ (41)}$$

Putting $a=hx-1,\ b = h\sqrt{x^2-1}$ and $a^2-b^2=1-2hx+h^2$ in (41)

We get $$\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{x^2-1}\cos\phi\right]h-1} = \pi\left(1-2xh+h^2\right)^{-\frac{1}{2}} \quad \text{--------- (44)}$$

If h is large so that $\left|\left\{x \pm \sqrt{x^2-1}\cos\phi\right\} h\right| > 1$, then both sides of (44) can be expanded in descending powers of h

(i.e.) $$\int_0^\pi \frac{d\phi}{t-1} = \frac{\pi}{h}\left[1-\frac{2x}{h}+\frac{1}{h^2}\right]^{\frac{-1}{2}}$$

where $t = \left\{x \pm \sqrt{x^2-1}\cos\phi\right\} h > 1$

(i.e.,) $$\int_0^\pi \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} d\phi = \frac{\pi}{h}\sum_{n=0}\frac{1}{h^n}P_n(x)$$

or $$\int_0^\pi \sum_{n=0}\frac{1}{t^{n+1}}d\phi = \pi \sum_{n=0}\frac{1}{h^{n+1}}P_n(x)$$

(i.e.) $$\frac{1}{\pi}\int_0^\pi \sum_{n=0}\frac{d\phi}{\left[\left\{x \pm \sqrt{x^2-1}\cos\phi\right\}h\right]^{n+1}} = \sum_{n=0}\frac{1}{h^{n+1}}P_n(x)$$

Equating the coefficients of $\frac{1}{h^{n+1}}$ on both sides, we get

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}} \quad \text{----------- (43)}$$

1.9 Christoffel's Expansion :

Show that

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n+5} + ... \quad \text{------------- (44)}$$

the last term being $3P_1$ or P_0 according as n is even or odd.

Proof : We have the recurrence relation

$$(2n+1)P_n = P_{n+1}' - P_{n-1}' \quad \text{-------------- (23)}$$

Similarly $$(2n-1)P_{n-1} = P_n' - P_{n-2}'$$

or $$P_n' = (2n-1)P_{n-1} + P'_{n-2}$$

$$=(2n-1)P_{n-1} + \left[(2n-5)P_{n-3} + P'_{n-4}\right]$$

$$=(2n-1)P_{n-1} + (2n-5)P_{n-3} + \left[(2n-9)P_{n-5} + P'_{n-6}\right]$$

$\therefore$ When n is even

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...P_2'$$

$$= (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...+3P_1 \quad \begin{pmatrix}\because P_0 = 1 \\ \therefore P_0' = 0\end{pmatrix}$$

the last term being $3P_1$.

Again when n is odd,

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...+ P_3'$$

$$=(2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...+(5P_2 + P_1')$$

$$=(2n-1)P_{n-1} + (2n-5)P_{n-3} + ...+ P_0 \quad (\because P_1 = x, P_1' = 1 = P_0)$$

thus the expansion (44) is proved.

(3) : Prove that

$$\int_{-1}^{1} (P_n')^2\, dx = n(n+1) \qquad \text{---------- (45)}$$

Solution : We have the Christoffel expansion

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ... \quad (44)$$

the last term being $3P_1$ or P_0 according as n is even or odd.

$$\therefore \int_{-1}^{1} (P_n')^2\, dx = (2n-1)^2 \int_{-1}^{1} (P_{n-1})^2\, dx + (2n-5)\int_{-1}^{1} (P_{n-3})^2\, dx + ...$$

the integral of the product of different P's vanishes in view of the orthogonal property (38)

So $\int_{-1}^{1} (P_n')^2\, dx = (2n-1)^2 \dfrac{2}{2(n-1)+1} + (2n-5)^2 \dfrac{2}{2(n-3)+1} + ...$ by equation (39).

$$= (2n-1)^2 \frac{2}{2n-1} + (2n-5)^2 \frac{2}{2n-5} + ...$$

$$= 2\left[(2n-1)+(2n-5)+.....\right]$$

In this, if n is even, the last term will be $\int_{-1}^{1} (3P_1)^2 dx$ which is equal to $9.\frac{2}{2.1+1} = 6$

So where n is even,

$$\int_{-1}^{1} (P_n')^2\, dx = 2\left[(2n-1)+(2n-5)+.....3\right]$$

This series is in AP of $\frac{n}{2}$ terms having first term $(2n-1)$ and last term 3 so that

$$\text{sum} = \frac{\text{no of terms}}{2} \text{ [1st term + last term]}$$

So $\int_{-1}^{1} (P_n')^2\, dx = 2.\frac{1}{2}\frac{n}{2}\left[(2n-1)+3\right] = n(n+1)$

Again when n is odd, we have

$$\int_{-1}^{1} (P_n')^2\, dx = 2\left[(2n-1)+(2n-5)+.......+1\right]$$

$$= 2\,\frac{1}{2}\,\frac{n+1}{2}\left[(2n-1)+1\right] = n(n+1)$$

Thus $\int_{-1}^{1} (P_n')^2\, dx = n(n+1)$ for all +ve integral values.

1.10 Associated Legendre Polynomials :

If the Laplace equation

$\nabla^2 V = \frac{\partial^2 \text{v}}{\partial x^2} + \frac{\partial^2 \text{v}}{\partial y^2} + \frac{\partial^2 \text{v}}{\partial z^2} = 0$ is transformed from cartesian coordinates to spherical polar coordinates by means of the relations $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, we obtain, after a lengthy but straight forward reduction.

$$r^2 \sin\theta \frac{\partial^2 v}{\partial r^2} + 2r \sin\theta \frac{\partial v}{\partial r} + \sin\theta \frac{\partial^2 v}{\partial \theta^2} + \cos\theta \frac{\partial v}{\partial \theta} + \frac{1}{\sin\theta} \frac{\partial^2 v}{\partial \phi^2} = 0 \quad \text{---------- (46)}$$

Any solution $V(r, \theta, \phi)$ of Laplace equation (46) is known as a Spherical Harmonic.

In an attempt to solve (46), we assume a product solution $V(r,\theta,\phi) = R(r)G(\theta, \phi)$. Substituting this in (46) and rearranging the terms, we get

$$\frac{r^2 R'' + 2r R'}{R} = -\left(\frac{1}{G} \frac{\partial^2 G}{\partial \theta^2} + \frac{Cos\theta}{G\, Sin\theta} \frac{\partial G}{\partial \theta} + \frac{1}{G\, Sin^2\theta} \frac{\partial^2 G}{\partial \phi^2} \right)$$

This relation can hold only if the common value of these two expressions is a constant. For the sake of convenience, we write the constant as $n(n+1)$. Thus we have

$$r^2 R'' + 2rR' - n(n+1)R = 0 \quad \text{-------- (47)}$$

$$\frac{\partial^2 G}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta} \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 G}{\partial \phi^2} + n(n+1)G = 0 \quad \text{---------- (48)}$$

Can interest lies with the solution of (48) which are known as **surface harmonics**.

In (48) we substitute $G(\theta,\phi) = G_1(\theta) G_2(\phi)$.

We find, after rearrangement of term,

$$\sin^2\theta \frac{G_1''(\theta)}{G_1(\theta)} + \sin\theta \cos\theta \frac{G_1'(\theta)}{G_1(\theta)} + n(n+1)\sin^2\theta = -\frac{G_2''(\phi)}{G_2(\phi)}$$

Taking the common constant value as m^2, we have pair of equations as

$$G_2'' + m^2 G_2 = 0 \quad \text{----------- (49)}$$

$$\sin^2\theta\ G_1' + \sin\theta \cos\theta\ G_1' + \left[n(n+1)\sin^2\theta - m^2 \right] G_1 = 0 \quad \text{------ (50)}$$

This equation (50) is known as the **associated Legendre Equation**. But the usual form can be obtained by putting $x = \cos\theta$. Thus on substitution and simplification, (50) can be written as

$$(1-x)^2 \frac{d^2 G_1}{dx^2} - 2x \frac{dG_1}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2} \right] G_1 = 0 \quad \text{------(51)}$$

which is the **algebric form of the associated Legendre differential equation.**

Further, putting $G_1(\theta) = (1-x^2)^{m/2} y$, the equation (51) is transformed to

$$(1-x^2)y'' - 2(m+1)xy' + [n(n+1) - m(m+1)]y = 0 \text{ -------- (52)}$$

On differentiating (52), we have

$$(1-x^2)y''' - [2x + 2(m+1)x]y'' + [-2(m+1) + n(n+1) - m(m+1)]y' = 0$$

or $$(1-x^2)y''' - 2(m+2)xy'' + [n(n+1) - (m+1)(m+2)]y' = 0 \text{ --------- (53)}$$

Obviously if y satisfies (52) for m, then $\frac{dy}{dx}$ satisfies (52) for (m + 1) as can be seen by (53).

Again for m = 0, (52) becomes Legendre's equation and hence $y = \frac{d^m P_n(x)}{dx^m}$ satisfies (52).

$$\therefore G_1(\theta) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$$

$= P_n^m(x)$ is the solution of (51)

or $$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \text{ ----------- (54)}$$

are called the **associated Legendre Polynomials** or **associated harmonics** of mth order and nth degree.

Equation (54) gives a relationship between the associated Legendre polynomial $P_n^m(x)$ and the Legendre Polynomial $P_n(x)$. This facilitates to derive recurrence relations and other relations for the associate Legendre polynomials.

(4) : Starting with the Legendre differential equation for $P_m(x)$, derive the differential equation for assocaited Legendre polynomials $P_m^n(x)$.

Solution : Legendre differential equation for $P_m(x)$ is

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \text{ -------- (55)}$$ where m is a +ve integer.

Differentiating this equation n times with the help of Leibnitz formula, we get

$$(1-x^2)u'' - 2x(n+1)u' + (m-n)(n+m+1)u = 0 \text{ ------ (56)}$$

where $u = \frac{d^n P_m(x)}{dx^n}$

Now Let $v(x) = (1-x^2)^{n/2} u(x)$

$$= (1-x^2)^{n/2} \frac{d^n P_m(x)}{dx^n}$$

Then we have

$$u' = \left(v + \frac{nxv}{1-x^2} \right)(1-x^2)^{-n/2} \text{ and}$$

$$u'' = \left(v'' + \frac{2nxv'}{1-x^2} + \frac{nv}{1-x^2} + \frac{n(n+2)x^2 v}{(1-x^2)^2} \right)(1-x^2)^{-n/2}$$

Substituting these values in (56), we get

$$(1-x^2)v'' - 2xv' + \left[m(m+1) - \frac{n^2}{1-x^2} \right] v = 0$$

Which is the Associated Legendre differnetial Equation

(5) : Express the electrostatic potential between two electric charges, at a distance d apart as a series of Legendre polynomials.

Solution : The associated electrostatic potential between two electric charges at the distance d is proportional to $\frac{1}{d}$ (i.e.,) $V = \frac{k}{d}$. Where k is an appropriate constant. Let the two charges have the position vectors $\hat{r}$ and $\hat{R}$ as shown in the Fig.1.

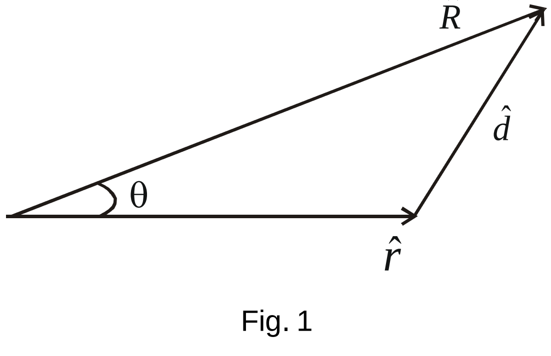


Fig. 1

The distance between them is

$$d = \left|\hat{R} - \hat{r}\right| = \left[R^2 - 2Rr\,Cos\,\theta + r^2\right]^{\frac{1}{2}}$$

$$= R\left[1 - \frac{2r}{R}Cos\,\theta + \left(\frac{r}{R}\right)^2\right]^{1/2} \quad \text{.............. (57)}$$

$$\therefore V = \frac{k}{R}\left[1 - \frac{2r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right]^{-1/2} \quad \text{---------- (58)}$$

For $\left|\hat{r}\right| = r < \left|\hat{R}\right|$, we may take the change of variables as $\frac{r}{R} = h$ and $\cos\theta = x$.

$\therefore (58)$ becomes

$$V = \frac{k}{R}\left[1 - 2xh + h^2\right]^{-1/2}$$

$$= \frac{k}{R}\sum P_n(x)h^n$$

Which is a series of Legendre functions known as the expansion of the generating function.

1.11 Summary :

The entire lesson consists of finding the solution in series of the Legendre differential equation and its properties. Prior to this, some basics to arrive at the power series solution of the differential equation are given.

The generating functions, from which a series of Legendre polynominals can be written, is given. With this, several recurrence relations are derived.

Then the differential representation (Rodrigue's formula), integral representation such as Laplace first integral and Laplace second integrals have been derived.

Orthonormalization codition of the Legendre polynominals and the Christoffd expansion of the derivative of the Legendre polynomial in a series of Legendre polynomials are derived.

The associated Legendre polynomials have been explained and its relationship with the Legendre polynomials is obtained.

1.12 Keyterminology :

Power series solution - analytic - singularity - Frobenius series - indicial equation - Legendre polynomial - Generating function - Rodrigue's formula - Laplace integral - Christoffel Expansion - Associatd Legendre polynomials

1.13 Self Assessment Questions :

1. (a) Show that $P_n(x)$ is an even function of x when n is even and is an odd function of x when n is odd.

   (b) Prove that $\left(1-x^2\right)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

2. (a) State and prove Rodrigue's formula.

   (b) Express x^4 as a series in Legendre polynomials.

3. (a) Prove $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$

   (b) Show that $\int_{-1}^{1} x\, P_n(x)\, P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$

4. (a) Show that $P_n(-x) = (-1)^n . P_n(x)$ (Hint : use generating function)

   (b) Prove that $\int_{-1}^{1} x^m\, P_n(x)\, dx = 0$ for m < n (Hint : use Rodrignes formula)

5. By differentiating the generating function for the Legendre polynomials, show that all even order derivatives of $P_n(x)$ vanish at x = 0, if n is odd and that all odd order derivatives vanish at x = 0 if n is even.

1.14 Reference Books :

1. B.D. Gupta - "Mathematical Physics" Vikas Publishing House, 1980
2. E. Kreyszig - "Advanced Engineering Mathematics" Wiley Eastern Pvt. Ltd., , 1971
3. P.P. Gupta, R.P.4S Yadav and G.S. Malik "Mathematical Physics" Kedarnath Ramnath, Meerut, 1980.

Unit - I

Lesson - 2
Bessel Functions

Objective of the lesson :

> To define beta and gamma functions.
> To state different forms and properties of beta and gamma functions.
> To find the series solution of Bessel differential equation.
> To find the generatingfunction, integral representation, recurrence relations of $J_n(x)$.
> To study the orthogonal properties and modified Bessel functions.

Structure of the lesson :

2.1 Introduction.
2.2 Definition of beta and gamma functions.
2.3 Fundamental property of gamma functions.
2.4 The value of $\Gamma\frac{1}{2}$ and graph of the gamma function.
2.5 Transformation of gamma function.
2.6 To show that $\beta(m,n) = \beta(n,m)$.
2.7 Different forms of beta function.
2.8 To find the relation between beta and gamma function.
2.9 Reduction of definite integrals to gamma functions.
2.10 Bessel's Equation, functions and polynomials.
[A] Series solution of Bessel's Differential equation.
[B] Generating function for $J_n(x)$.
[C] Integrals for $J_0(x)$ and $J_n(x)$.
[D] Recurrence formulae for $J_n(x)$.
[E] Examples.
[F] Orthogonal properties of Bessel's polynomials.
[G] Modified Bessel functions.
[H] Miscellaneous examples.

2.11 Summary of the lesson.
2.12 Key terminology.
2.13 Self Assessment questions.
2.14 Reference Books.

2.1. Introduction :

The most important of all variable coefficient differential equations is Bessel's differential equations. This arises in a great variety of problems, including almost all applications involving partial differential equations, such as the wave equation or the heat equation.

As a presequete to Bessel functions and many other forthcoming lessons, beta and gamma functions are to be dealt forthwith.

2.2. Definition of beta and gamma functions :

Under the study of Definite Integrals, we come across two very important integrals known as Eulerian Integrals which are of the type

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx \text{ and } \int_0^\infty e^{-x} x^{n-1} \, dx,$$

where the quantities m and n are supposed to be positive.

The first Eulerian integral is generally known as Beta Function and defined as

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$ where m and n are positive.

The second Eulerian integral is known as Gamma Function and is defined as

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$, where n is positive.

2.3. FUNDAMENTAL PROPERTY OF GAMMA FUNCTIONS :

$$\Gamma(n+1) = n\,\Gamma(n)$$

In order to prove this relation, let us consider the integral

$$\int_0^\infty e^{-x} x^n \, dx = \Gamma(n+1).$$

Integrating it by parts taking e^{-x} as second function, we get

$$\int_0^\infty e^{-x} x^n \, dx$$

$$= \left[-e^{-x} x^n\right]_0^\infty - n\int_0^\infty -e^{-x} x^{n-1} \, dx$$

$$= n\int_0^\infty -e^{-x} x^{n-1} \, dx,$$

$$\therefore \Gamma(n+1) = n\Gamma(n) \qquad \text{.......... (1)}$$

From (1) it is evident that if the value of $\Gamma(n)$ is known for n between two successive

3)

positive integers, then the value Γn for any positive value of n can be written as

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad \text{.......... (2)}$$

If -1<n<0 then (2) gives Γn, since n+1 is positive. As such, the value of Γn may be determined if -2<n<-1 since then $\Gamma(n+1)$ on the R.H.S. of (2) is known. Similarly Γn may be determined when -3<n<-2 and so on so forth.

Hence $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx = \frac{\Gamma(n+1)}{n}$ define Γn completely for all value of n except n = 0, -1, -2, -3,...........

Now replacing n by n-1 in (1) we get

$$\Gamma n = (n-1)\,\Gamma(n-1)$$

Similarly $\Gamma(n-1) = (n-2)\,\Gamma(n-2)$ etc.

Hence (1) yields

$$\Gamma(n+1) = n(n-1)(n-2)\text{.........} 3.2.1\ \Gamma(1)$$

But by definition $\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\text{........} 3.2.1 = \Gamma(n) \qquad \text{.......... (3)}$$

provided n is a positive integer

Putting n=0 in (3) we get

$$\Gamma(1) = 0! = 1 \qquad \because 0! = 1$$

$$\therefore \quad \Gamma(1) = 1 \qquad \text{...... (4)}$$

Also if we put n=0 in (2), then we find

$$\Gamma(0) = \frac{\Gamma(1)}{0} = \infty \qquad \text{...... (5)}$$

By repeated application of (2), it may be shown that the gamma function becomes infinite when x is zero or any negative integer i.e.,

$$\Gamma(-n) = \infty \qquad \text{...... (6)}$$

when n=0 or a positive integer.

But the function has finite value for negative values of n which are not integers.

2.4. THE VALUE OF $\Gamma\left(\frac{1}{2}\right)$ AND GRAPH OF THE GAMMA FUNCTION

We have by definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

Putting $x = \phi^2$ i.e., $dx = 2\phi\, d\phi$, we get

$$\Gamma(n) = 2\int_0^\infty \phi^{2n-1} e^{-\phi^2} d\phi$$

when $n = \frac{1}{2}$, this yields,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-\phi^2} d\phi \quad \text{......... (7)}$$

Suppose $I = \int_0^\infty e^{-\phi^2} d\phi$

Putting $\phi = \lambda\psi$ so that $d\phi = \lambda\, d\psi$

We have $I = \int_0^\infty e^{-\lambda^2\psi^2} . \lambda\, d\psi$

Multiplying both sides by $e^{-\lambda^2}$, we find

$$I . e^{-\lambda^2} = \int_0^\infty e^{-\lambda^2(1+\psi^2)} . \lambda\, d\psi$$

Integrating both sides w.r.t. λ within the limits 0 to ∞,

$$I \int_0^\infty e^{-\lambda^2} d\lambda = \int_0^\infty \int_0^\infty e^{-\lambda^2(1+\psi^2)} \lambda\, d\lambda\, d\psi$$

or
$$I \int_0^\infty e^{-\varphi^2} d\varphi = \int_0^\infty \left[-\frac{1}{2} . \frac{e^{-\lambda^2(1+\psi^2)}}{1+\psi^2} \right]_0^\infty d\psi$$

$$I^2 = \frac{1}{2} \int_0^\infty \frac{d\psi}{1+\psi^2} = \frac{1}{2}\left[\tan^{-1}\psi\right]_0^\infty = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4}$$

$$\therefore \quad I = \int_0^\infty e^{-\varphi^2}\, d\varphi = \frac{\sqrt{\pi}}{2} \qquad \text{.......... (8)}$$

From (7) and (8), we get

$$\Gamma\left(\frac{1}{2}\right) = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi} \qquad \text{.......... (9)}$$

Now putting $n = -\frac{1}{2}$ in (8), we find

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = -2\sqrt{\pi} \quad \text{by (9)} \qquad \text{.......... (10)}$$

Similarly $$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = -\frac{2}{3}(-2\sqrt{\pi}) = 4\frac{\sqrt{\pi}}{3} \text{ etc.} \qquad \text{.......... (11)}$$

The graph of Γn may be shown as below under the definition that the function becomes continuous function of n except when n=0 or any negative integer.

2.5. TRANSFORMATION OF GAMMA FUNCTION

By definition

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx \qquad \text{........ (12)}$$

Putting $x = \lambda y, \quad dx = \lambda\, dy$ in (12) we get

$$\Gamma(n) = \int_0^\infty e^{-\lambda y}\, \lambda^n\, y^{n-1}\, dy$$

or $$\frac{\Gamma(n)}{\lambda^n} = \int_0^\infty e^{-\lambda y}\, y^{n-1}\, dy. \qquad \text{.......... (13)}$$

If we put $e^{-x} = y$ in (12) then we get

$$\Gamma(n) = -\int_1^0 y\left(\log\frac{1}{y}\right)^{n-1} \frac{1}{y}\, dy = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \qquad \text{.......... (14)}$$

$$\left[\because \quad x = \log\frac{1}{y} \text{ and } dx = \frac{1}{1/y}\left(-\frac{1}{y^2}\right)dy = -\frac{1}{y}dy\right].$$

Again if we write $x = y^{1/n}$ in (12) we get

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{1/n}} (y)^{(n-1)/n}. (y)^{(1-n)/n} dy$$

$$\left[\because \quad dx = \frac{1}{n} y^{(1-n)/n} dy\right]$$

$$= \frac{1}{n}\int_0^\infty e^{-y^{1/n}} dy.$$

$$\therefore \quad n\,\Gamma(n) = \Gamma(n+1) = \int_0^\infty e^{-y^{1/n}} dy \qquad \text{......... (15)}$$

COROLLARY. If we replace n by $\frac{1}{2}$ in (15), we find

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} dy,$$

which is the same as (7).

2.6. TO SHOW THAT, $\beta(m,n) = \beta(n,m)$.

By definition

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

Replacing x by 1-x, we get

$$\beta(m,n) = \int_1^0 (1-x)^{m-1} \{1-(1-x)\}^{n-1} (-dx)$$

$$= \int_0^1 (x)^{n-1} (1-x)^{m-1} dx$$

$$= \beta(n,m).$$

2.7. DIFFERENT FORMS OF BETA FUNCTION

Substituting $\frac{y}{1+y}$ for x, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \left[\because dx = \frac{(1+y)-y}{(1+y)^2}dy = \frac{1}{(1+y)^2}dy \text{ and } 1-x = \frac{1}{1+y}\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \qquad \text{.......... (16)}$$

Also, since $\beta(m,n) = \beta(n,m)$,

$$\therefore \qquad \beta(m,n) \Rightarrow \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+m}}\,dy. \qquad \text{.......... (17)}$$

2.8. TO FIND THE RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}. \qquad \text{.......... (18)}$$

From equation (13), we have

$$\frac{\Gamma m}{\lambda^m} = \int_0^\infty e^{-\lambda x}\,x^{m-1}\,dx,$$

i.e,
$$\Gamma m = \int_0^\infty \lambda^m\,e^{-\lambda x}\,x^{m-1}\,dx.$$

Multiplying both sides by $e^{-\lambda}\lambda^{n-1}$ and integrating w.r.t. λ within the limits 0 to ∞, we get

$$\Gamma m \int_0^\infty e^{-\lambda}.\lambda^{n-1}\,d\lambda = \int_0^\infty\left[\int_0^\infty e^{-\lambda(1+x)}.\,\lambda^{m+n-1}\,d\lambda\right]x^{m-1}\,dx$$

or
$$\Gamma m.\Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\,.\,x^{m-1}\,dx \qquad \text{by equ. (13)}$$

$$= \Gamma(m+n)\,\beta(m,n) \qquad \text{by definition.}$$

$$\therefore \qquad \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.$$

COROLARRY. If we put $m+n = 1$ in

$$\Gamma m\,\Gamma n = \int_0^\infty \Gamma(m+n)\,.\,\frac{x^{m-1}}{(1+x)^{m+n}}\,dx,$$

we have $\Gamma m\,\Gamma(1-m) = \int_0^\infty \frac{x^{m-1}}{1+x}\,dx \qquad [\because\ \Gamma 1 = 1]$

$$= \frac{\pi}{\sin m\pi} \quad \text{(standard integral)} \qquad \text{.......... (19)}$$

Replacing m by $\frac{1}{2}$, we get

$$\Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \pi \quad \text{or} \quad \Gamma\frac{1}{2} = \sqrt{\pi}.$$

2.9. REDUCTION OF DEFINITE INTEGRALS TO GAMMA FUNCTIONS

[1] To show that

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

we know that

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$

Substituting $\frac{1}{x}$ for y in the second integral on R.H.S., we get

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_1^0 \frac{\left(\frac{1}{x}\right)^{m-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{1}{x^2}\right)dx,$$

$$\therefore\ y = \frac{1}{x}, \quad dy = -\frac{1}{x^2}\,dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\therefore \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

[∵ change of variable variable does not change the value of integral]

or $$\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \qquad \text{........ (20)}$$

[2] If we substitute $\frac{ay}{b}$ for x, we get

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = a^m b^n \int_0^\infty \frac{y^{m-1}\,dy}{(ay+b)^{m+n}}$$

Since $\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ by (16) and (18).

$$\therefore \quad a^m\,b^n \int_0^\infty \frac{y^{m-1}\,dx}{(ay+b)^{m+n}} = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

or $$\int_0^\infty \frac{y^{m-1}\,dx}{(ay+b)^{m+n}} = \frac{\Gamma m\,\Gamma n}{a^m\,b^n\,\Gamma(m+n)}.$$

COROLLARY. Substituting $y = \tan^2\theta$, this relation transforms to

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma m\,\Gamma n}{2\,a^m\,b^n\,\Gamma(m+n)}.$$

[3] If we put $x = \sin^2\theta$, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2(m-1)}\theta\,\cos^{2(n-1)}\theta\,\sin\theta\,\cos\theta\,d\theta$$

$$(\because \quad dx = 2\sin\theta\cos\theta\,d\theta)$$

or $$\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,\sin\theta\,\cos\theta\,d\theta$$

$$\therefore \quad \int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma(m+n)} \qquad \text{........ (21)}$$

COROLLARY. Replacing 2m-1 by p and 2n-1 by q, this relation reduces to $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$ .... (22)

Putting p=0 and q=0 in succession, we get

$$\int_0^{\pi/2} (\cos\theta)^q \, d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\sqrt{\pi}}{2\,\Gamma\left(\frac{q}{2}+1\right)} \quad \text{and} \quad \int_0^{\pi/2} (\sin\theta)^p \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma\left(\frac{p}{2}+1\right)} \cdot \frac{\sqrt{\pi}}{2}.$$

[4] By putting $x = \sin^2\theta$ in $\dfrac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^1 x^{m-1}(1-x)^{n-1} \, d\theta$.

We have just proved that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\,\Gamma(m+n)}. \qquad \text{............. (21)}$$

Now if we put 2n=1, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma m}{2\,\Gamma\left(m+\frac{1}{2}\right)} \qquad \text{............ (23)}$$

Again putting m=n in (1), we find

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

or $\dfrac{\{\Gamma(m)\}^2}{2\,\Gamma(2m)} = \dfrac{1}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$ $[\because \quad 2\sin\theta\cos\theta = \sin 2\theta]$

$$= \frac{1}{2^{2m}} \int_0^{\pi} \sin^{2m-1}\varphi \, d\varphi \quad \text{put } 2\theta = \varphi, \quad d\theta = \frac{1}{2}d\varphi$$

$$= \frac{1}{2^{2m}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\varphi \, d\varphi \qquad [\because \quad \sin(\pi-\varphi) = \sin\varphi]$$

(Prop. of definite integral)

$$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{2^{2m-2}(\Gamma m)^2}{\Gamma(2m)} \qquad \text{............. (24)}$$

From (23) and (24) it is obvious that

$$\frac{2^{2m-2}\ (\Gamma \Gamma m^{2}}{\Gamma(2m)} = \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)} \frac{\sqrt{\pi}}{2}$$

or $$\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\, \Gamma(2m).$$

This may also be put in the form

$$\Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}}\, \Gamma m\, \Gamma\left(m+\frac{1}{2}\right) \qquad \text{.............. (25)}$$

2.10. BESSEL'S EQUATION, FUNCTIONS AND POLYNOMIALS

[A] Series Solution of Bessel's Differential Equation.

This equation is of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1-\frac{n^2}{x^2}\right)y = 0. \qquad \text{............ (26)}$$

There is singularity at x=0, and this is a removable singularity and hence the given equation may be solved by the method of series integration.

In order to integrate it in a series of ascending powers of x, let us assume that its series solution is

$$y = \sum_{r=0}^{\infty} a_r\, x^{k+r}.$$

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r)\, x^{k+r-1},$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)\, x^{k+r-2}$$

Substituting these values in (26), we get

$$\sum_{r=0}^{\infty}\left[(k+r)(k+r-1)\, x^{k+r-2} + \frac{1}{x}(k+r)\, x^{k+r-1} + \left(1-\frac{n^2}{x^2}\right)x^{k+1}\right]a_r \equiv 0$$

or $$\sum_{r=0}^{\infty}\left[\{(k+r)(k+r-1) + (k+r) - n^2\}\, x^{k+r-2} + x^{k+r}\right]a_r \equiv 0$$

or $$\sum_{r=0}^{\infty}\left[\{(k+r)^2 - n^2\}\, x^{k+r-2} + x^{k+r}\right] a_r \equiv 0 \qquad \text{........... (27)}$$

The relation (27) being an identity, let us equate the coefficients of various powers r=0 in (27), we have the indicial equation

$$(k^2 - n^2)\, a_0 = 0.$$

Being the coefficient of first term, $a_0 \neq 0$.

$$\therefore \; k^2 - n^2 = 0, \;\; \text{i.e.,} \;\; k = \pm n. \qquad \text{........... (28)}$$

Now equating to zero the coefficient of x^{k-1} by putting r=1 in (27), we get

$$\{(k+1)^2 - n^2\}\, a_1 = 0$$

But from (28), $(k+1)^2 - n^2 \neq 0$; $\therefore a_1 = 0$ .... (29)

Equating to zero the coefficient of general term, i.e., x^{k+r} in (27), we find

$$\{(k+r+2)^2 - n^2\}\, a_{r+2} + a_r = 0$$

or $$a_{r+2} = -\frac{a_r}{(k+r+2-n)(k+r+2+n)} \qquad \text{............ (30)}$$

Case I. When k=+n. By putting r=0, 1, 2,......... in (30), we get

$$a_2 = -\frac{a_0}{2\,(2n+2)}$$

and $a_1 = a_3 = a_5 .... = 0$,

$$a_4 = -\frac{a_2}{4\,(2n+4)} = \frac{a_0}{2.4\,(2n+2)\,(2n+4)},$$

$$a_6 = -\frac{a_4}{6\,(2n+6)} = -\frac{a_0}{2.4.6\,(2n+2)\,(2n+4)\,(2n+6)}$$

....

$$a_{2r} = -\frac{(-1)^r\, a_0}{2.4.6......2r\,(2n+2)\,(2n+4)........(2n+2r)}.$$

Hence the series solution is

$$y = a_0 \left[x^n - \frac{x^{n+2}}{2\,(2n+2)} + \frac{x^{n+1}}{2.4\,(2n+2)\,(2n+4)} - \dots\dots\dots \right]$$

$$= a_0\, x^n \left[1 - \frac{x^2}{2\,(2n+2)} + \frac{x^4}{2.4\,(2n+2)\,(2n+4)} - \dots\dots \frac{(-1)^r\, x^{2r}}{2.4\dots\dots 2r\,(2n+2)\dots.(2n+2r)} + \dots\dots \right]$$

$$= a_0\, x^n \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2r}}{2^r\,(r)!\;2^r\,(n+1)\dots\dots(n+r)} \qquad \dots\dots\dots (31)$$

If $a_0 = \dfrac{1}{2^n\,\Gamma(n+1)}$, this solution is called as $J_n(x)$.

Thus $\quad J_n(x) = -\dfrac{x^n}{2^n\,\Gamma(n+1)} \cdot \displaystyle\sum_{r=0}^{\infty} \dfrac{(-1)^r\, x^{2r}}{2^{2r}\,(r)!\,(n+1)\,(n+2)\dots\dots(n+r)}$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{(r)!\;\Gamma(n+r+1)} \qquad \dots\dots\dots\dots (32)$$

Case II. When $k = -n$.

The series solution is obtained by replacing n by $-n$ in the value of $J_n(x)$, hence, we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r . \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)} \qquad \dots\dots\dots\dots\dots (33)$$

The completion primitive of Bessel's equation is

$$A\, J_n(x) + B\, J_{-n}(x),$$

where n is not an integer, A, B being two arbitrary constants.

COROLLARY. Bessel's equation for $n=0$ is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

Its series solution by the same substitution $y = \sum_{r=0}^{\infty} a_r\, x^{k+r}$ (as above) is obtained to be

$$y = a_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots\dots\dots \right).$$

If $a_0 = 1$, this solution is denoted by $J_0(x)$, i.e.,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \text{.........} \qquad \text{............ (34)}$$

where $J_0(x)$ is called Bessel function of *zeroeth order.*

In fact $J_0(x)$ is that solution of Bessel's equation for n=0, which is equal to unity for x=0.

Note : $J_n(x)$ is called Bessel's function of the first kind of order n.

[B] Generating Function for $J_n(x)$:

To show that $e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

Proof : We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{.......}$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

$$\therefore \; e^{xt/2} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r \, r!} \qquad \text{............ (35)}$$

Similarly, $e^{-x/2t} = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{2^s \, t^s \, s!}$ .... (36)

Multiplying (35) and (36), we get

$$e^{x/2\,(t-1/t)} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r r!} \times \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{2^s t^s s!}$$

In order to find the (t^n)th term, we should replace r by n+s and then coefficient of t^n

$$\sum_{s=0}^{\infty} \frac{x^{n+s}}{2^{n+s}(n+s)!} \times \frac{-(1)^s x^s}{2^s s!} = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{(n+s)!\, s!} = J_n(x) \qquad \text{........... (37)}$$

Again the coefficient of t^{-n} is obtained by putting s=n+r and then coefficient of $t^{-n} = \sum_{r=0}^{\infty} \frac{x^r}{2^r . r!} \times \frac{(-1)^{n+r} x^{n+r}}{2^{n+r}(n+r)!}$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!(n+r)!}$$

$$= (-1)^n . J_n(x)$$

$$= J_{-n}(x) \qquad \text{............. (38)}$$

since $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

It may be shown as below :

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)},$$

which tends to zero if -n+r+1 = 0, i.e., r = n-1 $(\because \Gamma 0 = \infty)$.

Hence all the terms upto nth, vanish and therefore the limit $r = 0$ may be changed to $r = n$.

$$\therefore \quad J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)}$$

Now putting r = n+s, where s is a positive integer, we have

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{n+s} \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{(n+s)!\,\Gamma(-n+n+s+1)}$$

$$= \sum_{s=0}^{\infty} (-1)^{n+s} \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{(n+s)!\,\Gamma(s+1)}$$

or $$J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{s!\,\Gamma(n+s+1)}$$

$$= (-1)^n J_n(x). \qquad \text{............. (39)}$$

Hence from (37) and (38), we have

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad \text{............. (40)}$$

COROLLARY. Putting $t = e^{i\phi}$ and $\frac{1}{t} = e^{-i\phi}$, we get

$$\text{Exp}\left(ix\,\frac{(e^{i\phi}-e^{-i\phi})}{2i}\right) = \sum_{n=-\infty}^{\infty} e^{ni\phi} J_n(x)$$

or $$e^{ix\sin\phi} = J_0(x) + \{J_1(x)\,e^{i\phi} + J_{-1}(x)\,e^{-i\phi}\} + \{J_2(x)\,e^{2i\phi} + J_{-2}(x)\,e^{-2i\phi}\} + \ldots\ldots$$

or $$\cos(x\sin\phi) + i\sin(x\sin\phi) = J_0(x) + J_1(x)\{e^{i\phi} - e^{-i\phi}\} + J_2(x)\{e^{2i\phi} + e^{-2i\phi}\} + \ldots\ldots$$

[since $J_n(x) = J_{-n}(x)$ when n is even]

$$= J_0(x) + 2i\sin\phi\, J_1(x) + 2\cos 2\phi\, J_2(x) + \ldots\ldots$$

Equating real and imaginary parts, we get

$$\cos(x\sin\phi) = J_0(x) + 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots\ldots \qquad \ldots\ldots (41)$$

and $$\sin(x\sin\phi) = 2\sin\phi\, J_1(x) + 2\sin 3\phi\, J_3(x) + 2\sin 5\phi\, J_5(x) + \ldots\ldots \qquad \ldots\ldots (42)$$

Replacing ϕ by $\frac{\pi}{2} - \phi$, we have, from (41) and (42)

$$\cos(x\cos\phi) = J_0(x) - 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots\ldots \qquad \ldots\ldots (43)$$

and $$\sin(x\cos\phi) = 2\cos\phi\, J_1(x) - 2\cos 3\phi\, J_3(x) + 2\cos 5\phi\, J_5(x) + \ldots\ldots \qquad \ldots\ldots (44)$$

C] **Integrals for $J_0(x)$ and $J_n(x)$:**

I. $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi$.

We have by (41),

$$\cos(x\sin\phi) = J_0(x) + 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots\ldots$$

If we integrate both the sides of this relation with respect to ϕ from the limits o to ϕ, then we see that all the integrals except first of the R.H.S. vanish, thereby giving

$$\int_0^{\pi} \cos(x\sin\varphi)\,d\varphi = J_0(x)\int_0^{\pi} d\varphi = \pi J_0(x) \qquad \ldots\ldots\ldots (45)$$

II. $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\,d\phi$

We have already proved that

$$\cos(x \sin\phi) = J_0(x) + 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots\ldots\ldots \qquad \ldots\ldots\ldots (46)$$

and $$\sin(x\sin\phi) = 2J_1(x)\sin\phi + 2\sin 3\phi\, J_3(x) + 2\sin 5\phi\, J_5(x) + \ldots\ldots\ldots \qquad \ldots\ldots\ldots (47)$$

If we multiply (46) by $\cos n\phi$, (47) by $\sin n\phi$ and integrate between the limits 0 to π, we have

$$\int_0^{\pi} \cos(x\sin\phi)\cos n\phi\, d\phi = 0 \quad \text{or} \quad \pi J_n(x) \qquad \ldots\ldots\ldots (48)$$

according as n is odd or even

and $$\int_0^{\pi} \sin(x\sin\phi)\sin n\phi\, d\phi = \pi J_n(x) \quad \text{or} \quad 0 \qquad \ldots\ldots\ldots (49)$$

according as n is odd or even.

Adding (48) and (49), we find

$$\int_0^{\pi} [\cos(x\sin\phi)\cos n\phi + \sin(x\sin\phi)\sin n\phi]\, d\phi = \pi J_n(x)$$

whether n is odd or even

or $$\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi = \pi J_n(x),$$

i.e., $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - x\sin\varphi)\, d\varphi \qquad \ldots\ldots\ldots (50)$$

III.

$$J_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x\sin\varphi)\cos^{2n}\varphi\, d\varphi \qquad \ldots\ldots\ldots (51)$$

If we expand $\cos(x\sin\varphi)$ in the powers of $x\sin\phi$, the general

term is $$(-1)^r \frac{x^{2r}}{(2r)!}\sin^{2r}\phi.$$

General term of R.H.S. of (51)

$$= \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n (-1)^r \int_0^{\pi} \frac{x^{2r}}{(2r)!}\sin^{2r}\phi\cos^{2n}\phi\, d\phi \qquad \ldots\ldots\ldots (52)$$

where $\int_0^{\pi} \sin^{2r}\phi \cos^{2n}\phi \, d\phi$

$$= 2 \int_0^{\pi/2} \sin^{2r}\varphi \cos^{2n}\varphi \, d\varphi$$

$$= \int_0^1 t^r (1-r)^n \frac{dt}{t^{1/2}(1-t)^{1/2}}$$

$$= \int_0^1 t^{(2r-1)/2} (1-t)^{(2n-1)/2} \, dt$$

Put $\sin^2\phi = t$,
$\therefore \quad 2\sin\phi\cos\phi \, d\phi = t$
or $\quad d\phi = \dfrac{dt}{2\sqrt{\{t(1-t)\}}}$

$$= \beta\left(\frac{2r+1}{2}, \frac{2n+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left\{\left(\frac{2r+1}{2}\right)+\left(\frac{2n+1}{2}\right)\right\}} \qquad \because \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$= \frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\ldots\ldots\frac{1}{2}\sqrt{\pi}\,.\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)}.$$

Substituting this value in R.H.S. of (52) we have

$$= \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n (-1)^r \int_0^{\pi} \frac{x^{2r}}{(2r)!} \sin^{2r}\phi \cos^{2n}\phi \, d\phi$$

$$= \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n (-1)^r \frac{x^{2r}}{(2r)!} \times \frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\ldots\ldots\frac{1}{2}\sqrt{\pi}\,.\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)}$$

$$= 2\left(\frac{x}{2}\right)^n (-1)^r . \frac{x^{2r}}{\Gamma(n+r+1)} \cdot \frac{(2r-1)(2r-3)\ldots\ldots 1}{2^r . 2r(2r-1)(2r-2) . 1}$$

$$= \left(\frac{x}{2}\right)^{n+2r} (-1)^r . \frac{1}{r!\;\Gamma(n+r+1)}$$

$$= (-1)^r \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{r!\,(n+r)!} = \text{general term in } J_n(x).$$

Hence $$J_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x \sin\varphi) \cos^{2n}\varphi \, d\varphi$$

.... (53)

[D] Recurrence Formulae for $J_n(x)$.

I. We know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^n \frac{1}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r},$$

where n is a positive integer.

Differentiating it w.r.t. x, we get

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}.$$

Multiplying both sides by x, we have

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r} + x \sum_{r=0}^{\infty} (-1)^r \frac{r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

or $$x \cdot J_n'(x) = n J_n(x) + x \sum_{r=1}^{\infty} (-1)^r \cdot \frac{1}{(r-1)!\,(n+r)!} \cdot \left(\frac{x}{2}\right)^{n-1+2r}$$

[since on R.H.S. the second term vanishes for r = 1 and hence limit of r = 0 may be replaced by r = 1

Putting r-1 = s, we have

$$x \cdot J_n'(x) = n J_n(x) + x \sum_{s=0}^{\infty} (-1)^{s+1} \cdot \frac{1}{s!\,(n+1+s)!} \cdot \left(\frac{x}{2}\right)^{n+2s-1}$$

$$= n J_n(x) - x \, J_{n+1}(x).$$ .... (54)

II. Again $x \cdot J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+2r)}{r!\,(n+r)!} \cdot \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right)$ may be written as

$$x \cdot J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{-n+2(n+r)}{r!\,(n+r)!} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{r!\,(n+r)!} \cdot \left(\frac{x}{2}\right)^{n+2r} + x \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{r!\,(n+r-1)!} \cdot \left(\frac{x}{2}\right)^{n-1+2r}$$

$$= -n\, J_n(x) + x\, J_{n-1}(x) \qquad \text{.......... (55)}$$

sum and difference of (54) ans (55) give

III. $2\, J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ .... (56)

IV. $2\, n\, J_n(x) = x\, \{J_{n+1}(x) + J_{n-1}(x)\}$ .... (57)

V. $\frac{d}{dx}\{x^n\, J_n(x)\} = x^n\, J_{n-1}(x)$ .... (58)

Here $\frac{d}{dx}\{x^n\, J_n(x)\} = n\, x^{n-1} J_n(x) + x^n J_n'(x)$

$$= x^{n-1}\{n\, J_n(x) + x J_n'(x)\}$$

$$= x^{n-1}\{n\, J_n(x) - n\, J_n(x) + x\, J_{n-1}(x)\} \quad \text{by (30)}$$

$$= x^n\, J_{n-1}(x).$$

VI. Similarly it is easy to show that

$$\frac{d}{dx}\{x^{-n}\, J_n(x)\} = -x^{-n}\, J_{n+1}(x). \qquad \text{........ (59)}$$

[E] Examples :

Ex (1) Show that $y = \frac{1}{\pi}\int_0^{\pi} \cos(x \cos\phi)\, d\phi$ satisfies the differential equation

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$ and that y is no other than $J_0(x)$.

Soln : Given $y = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\, d\phi$ .... (60)

If we differentiate it w.r.t. x under the sign of integration, we find

$$\frac{dy}{dx} = \frac{1}{\pi}\int_0^{\pi} -\cos\phi \sin(x\cos\phi)\, d\phi \quad \text{........ (61)}$$

and $$\frac{d^2y}{dx^2} = \frac{1}{\pi}\int_0^{\pi} -\cos^2\phi \cos(x\cos\phi)\, d\phi \quad \text{......... (62)}$$

Now from (61), we have

$$-\frac{dy}{dx} = \frac{1}{\pi}\left[\{\sin(x\cos\phi).\sin\phi\}_0^{\pi} - \int_0^{\pi} -x\sin\phi\cos(x\cos\phi).\sin\phi\, d\phi\right]$$

(integrating by parts)

$$= \frac{x}{\pi}\int_0^{\pi} \sin^2\phi \cos(x\cos\phi)\, d\phi$$

$$= \frac{x}{\pi}\int_0^{\pi} (1-\cos^2\phi)\cos(x\cos\phi)\, d\phi$$

$$= \frac{x}{\pi}\int_0^{\pi} \cos(x\cos\phi)\, d\phi - \frac{x}{\pi}\int_0^{\pi} \cos^2\phi\cos(x\cos\phi)\, d\phi$$

$$= xy + x\frac{d^2y}{dx^2} \quad \text{from (60) and (62)}$$

or $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

Hence $y = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\, d\phi$ satisfies $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$, which is the Bessel's equation for n=0.

Since y=1 when x=0, therefore y is no other than $J_0(x)$, as $J_0(x)$, being the solution of Bessel's equation, is unity for x=0.

Ex (2) : Prove that $J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x$.

Soln : We know that

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2.2\,(n+1)} + \frac{x^4}{2.4.2^2\,(n+1)\,(n+2)} \text{.........}\right].$$

$$\therefore \quad J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \cdot \left[1 - \frac{x^2}{2.2.\frac{3}{2}} + \frac{x^4}{2.4.2.\frac{3}{2}.\frac{5}{2}} \cdots\cdots\right]$$

$$= \sqrt{\left(\frac{x}{2}\right)} \frac{1}{\frac{1}{2}\sqrt{\pi}} \cdot \left[1 - \frac{x^3}{2.3} + \frac{x^4}{2.3.4.5} \cdots\cdots\right]$$

$$= \sqrt{\left(\frac{2x}{\pi}\right)} \frac{1}{x} \cdot \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)} \sin x .$$

Ex (3) : **Show that**

(i) $J_0'(x) = -J_1(x)$ and (ii) $2J_0''(x) = J_2(x) - J_0(x)$.

Soln : **From recurrence formula (54) we have**

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) .$$

Putting n=0, we get $J_0'(x) = -J_1(x)$,

which proves the first result.

Now differentiating it and multiplying throughtout by 2, we get

$$2 J_0''(x) = -2 J_1'(x)$$

or $$2 J_0''(x) = -\left[J_0(x) - J_2(x)\right]$$

by recurrence formula **III.** (56).

$$= J_2(x) - J_0(x) .$$

Ex (4) : **Prove that** $J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4) J_{n+4}$.

Soln : **From recurrenec formula IV, we have**

$$2 n J_n = x(J_{n-1} + J_{n+1}).$$

Replacing n by n+1, we get

$$\frac{2}{x}(n+4) J_{n+4} = J_{n+3} + J_{n+5} .$$

Ex (5) : Prove that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) \cdots\cdots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) \cdots\cdots$$

Soln : **We know that**

$$\cos(x \cos\phi) = J_0(x) - 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) \cdots\cdots \qquad \cdots\cdots\cdots\cdots (43)$$

and $$\sin(x \cos\phi) = 2\cos\phi\, J_1(x) - 2\cos 3\phi\, J_3(x) + 2\cos 5\phi\, J_5(x) \cdots\cdots \qquad \cdots\cdots\cdots\cdots (44)$$

Putting $\phi=0$, we get

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) \ldots\ldots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) \ldots\ldots$$

Ex (6) : Establish the relation

$$J_n(x)\,J'_{-n}(x) - J'_n(x)\,J_{-n}(x) = -\frac{2\sin n\pi}{\pi x}.$$

Soln : We know that $J_n(x)$ and $J_{-n}(x)$ are the two solutions of Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \qquad \ldots\ldots\ldots\ (26)$$

Hence, if $y = J_n(x)$, $\frac{dy}{dx} = J'_n(x)$ and $\frac{d^2y}{dx^2} = J''_n(x)$, we have from (26)

$$J''_n(x) + \frac{1}{x}J'_n(x) + \left(1 - \frac{n^2}{x^2}\right)J_n(x) = 0 \qquad \ldots\ldots\ldots\ (63)$$

Similarly putting $y = J_{-n}(x)$, we get from (26)

$$J''_{-n}(x) + \frac{1}{x}J'_{-n}(x) + \left(1 - \frac{n^2}{x^2}\right)J_{-n}(x) = 0 \qquad \ldots\ldots\ldots\ (64)$$

Multilying (63) by $J_{-n}(x)$, (64) by $J_n(x)$ and then subtracting (64) from (63), we have

$$J''_n(x)\,J_{-n}(x) - J''_{-n}(x)\,J_n(x) + \frac{1}{x}\left\{J'_n(x)\,J_{-n}(x) - J'_{-n}(x)\,J_n(x)\right\} = 0$$

Put $z = J'_n(x)\,J_{-n}(x) - J'_{-n}(x)\,J_n(x)$.

$$z' = J''_n(x)\,J_{-n}(x) + J'_n(x)\,J'_{-n}(x) - J'_{-n}(x)\,J'_n(x) - J''_{-n}(x)\,J_n(x)$$

$$= J''_n(x)\,J_{-n}(x) - J''_{-n}(x)\,J_n(x).$$

Thus $z' + \frac{z}{x} = 0$ or $\frac{z'}{z} = -\frac{1}{x}$.

Integrating, $\log z = -\log x + \log C$, where C is some arbitraty constant.

$$= \log\frac{C}{x}.$$

$$\therefore\ z = \frac{C}{x}.$$

or $\quad J'_n(x)\,J_{-n}(x) - J'_{-n}(x)\,J_n(x) = \frac{C}{x}.$

Equating the coefficients of $\frac{1}{x}$ on either side, we get

$$\frac{1}{2^n \Gamma(n+1)} \cdot \frac{1}{2^{-n} \Gamma(-n+1)} \{n-(-n)\} = C$$

or
$$C = \frac{2n}{\Gamma(n+1)\Gamma(-n+1)} = \frac{2}{\Gamma n \, \Gamma(1-n)}$$

$$= \frac{2}{\pi/\sin n\pi} = \frac{2\sin n\pi}{\pi}; \qquad \because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{..... (19)}$$

(Gamma functions).

Hence $J'_n(x) J_{-n}(x) - J'_{-n}(x) J_n(x) = \frac{2\sin n\pi}{\pi}$.

[F] Orthogonal Properties of Bessel's Polynomials.

To prove that $\int_0^a J_n(\mu r) J_n(\mu' r)\, r\, dr = 0$ where μ and μ' are different roots of $J_n(\mu a) = 0$.

Since $J_n(x)$ is a solution of Bessel's equation,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad \text{......... (26)}$$

therefore putting $x = \mu r$ and calling $y = u$ in (26) we get

$$\frac{1}{\mu^2}\frac{d^2\mu}{dr^2} + \frac{1}{\mu r}\frac{1}{\mu}\frac{du}{dr} + \left(1 - \frac{n^2}{\mu^2 r^2}\right) u = 0,$$

$$\left[\because \frac{dy}{dx} = \frac{du}{dx} = \frac{du}{dr}\cdot\frac{dr}{dx} = \frac{1}{\mu}\frac{du}{dr} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{\mu}\frac{d}{dr}\left(\frac{1}{\mu}\frac{du}{dr}\right) = \frac{1}{\mu^2}\frac{d^2u}{dr^2}\right].$$

Multiplying throughout by $\mu^2 r^2$,

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + (\mu^2 r^2 - n^2)\, u = 0 \qquad \text{........... (65)}$$

Similarly putting $x = \mu' r$ and calling $y = v$ in (36), we have

$$r^2\frac{d^2v}{dr^2} + r\frac{dv}{dr} + (\mu'^2 r^2 - n^2)\, v = 0 \qquad \text{........... (66)}$$

If we multiply (65) by $\frac{v}{r}$, (66) by $\frac{u}{r}$ and subtract

$$(r.vu'' - ruv'') + (vu' - uv') + (\mu^2 - \mu'^2)\, rvu = 0,$$

where $u' = \dfrac{du}{dr}$ and $v' = \dfrac{dv}{dr}$ etc.

or
$$\frac{d}{dr} r\{(vu' - uv')\} + [(\mu^2 - \mu'^2)]\, rvu = 0, \quad \text{.......... (67)}$$

where $u = J_n(\mu r)$ and $v = J_n(\mu' r)$

Integrating (67) w.r.t r between the limits 0 and a, we get

$$[r\{J_n(\mu r) . J_n'(\mu' r)\, \mu' - J_n(\mu' r)\, J_n'(\mu r)\, \mu\}]_0^a$$
$$+ \int_0^a (\mu^2 - \mu'^2)\, J_n(\mu r)\, J_n(\mu' r)\, r\, dr = 0$$

The first term vanishes for both the limits since

$$J_n(\mu a) = 0\ ;\ J_n(\mu' a) = 0$$

Hence $\displaystyle\int_0^a (\mu^2 - \mu'^2)\, J_n(\mu r)\, J_n(\mu' r)\, r\, dr = 0$

i.e., $\displaystyle\int_0^a J_n(\mu r)\, J_n(\mu' r)\, r\, dr = 0 \quad \because \mu^2 - \mu'^2 \neq 0$ .... (68)

[G] Modified Bessel functions :

Bessel's differential equation of n th order is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \quad \text{.............. (26)}$$

Put $x = iz$ so that $\dfrac{dx}{dz} = i$ or $\dfrac{dz}{dx} = \dfrac{1}{i}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} . \frac{dz}{dx} = \frac{1}{i}\frac{dy}{dz} \quad \text{and}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{i}\frac{dy}{dz}\right) = \frac{1}{i} . \frac{d^2y}{dx^2} . \frac{dz}{dx} = \frac{1}{i^2}\frac{d^2y}{dz^2} = -\frac{d^2y}{dz^2}$$

with these substitution, equ. (26) becomes

$$-\frac{d^2y}{dz^2} + \frac{1}{iz} . \frac{1}{i} . \frac{dy}{dz} + \left(1 + \frac{n^2}{z^2}\right) y = 0$$

or
$$\frac{d^2y}{dz^2} + \frac{1}{z} . \frac{dy}{dz} - \left(1 + \frac{n^2}{z^2}\right) y = 0 \quad \text{................ (69)}$$

This is called modified Bessel equation.
The modified Bessel function is obtained by putting $x = iz$ in the function

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r} \qquad \text{.............. (32)}$$

$$\therefore \quad J_n(iz) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r+1)!} \left(\frac{z}{2}\right)^{n+2r} \cdot i^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{i^n.(-1)^{2r}}{r!\,(n+r+1)!} \left(\frac{z}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{i^n}{r!\,(n+r+1)!} \left(\frac{z}{2}\right)^{n+2r}$$

or $$i^{-n} J_n(iz) = \sum_{r=0}^{\infty} \frac{1}{r!\,(n+r+1)!} \left(\frac{z}{2}\right)^{n+2r}$$

or $$I_n(x) = i^{-n} J_n(ix) = \sum_{r=0}^{\infty} \frac{1}{r!\,(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r} \qquad \text{............. (70)}$$

is called the modified Bessel function of the first kind of order n.

[H] Miscellaneous Examples :

(1) Show that $2^n\, \Gamma\left(n+\frac{1}{2}\right) = 1.3.5........ (2n-1)\sqrt{\pi}$.

Soln : We know that $\Gamma(n+1) = n\Gamma n$

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

- - - - - - - -

- - - - - - - -

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)........\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}........\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}$$

$$= \frac{1.3.5............(2n-5)(2n-3)(2n-1)}{2^n}.\sqrt{\pi}$$

or $$2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1.3.5..........(2n-1)\sqrt{\pi}.$$

(2) Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} \cdot \frac{\sqrt{\pi}}{n}$.

Soln : Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

Put $x^n = \sin^2\theta \qquad \therefore\ x = \sin^{2/n}\theta$

$$dx = \frac{2}{n} \cdot \sin^{\frac{2-n}{n}}\theta \, \cos\theta \, d\theta$$

so $$I = \frac{2}{n} \cdot \int_0^{\pi/2} \frac{\sin^{\frac{2-n}{n}}\theta \, \cos\theta}{\cos\theta} d\theta$$

$$= \frac{2}{n} \cdot \int_0^{\pi/2} \sin^{\frac{2-n}{n}}\theta \, d\theta$$

$$= \frac{2}{n} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{n}\right)}{2\,\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} \cdot \frac{\sqrt{\pi}}{n} \qquad \text{from (22)}$$

(3) Show that the beta functions satisfy the following relation

$$\beta(a,b) \,.\, \beta(a+b, c) = \beta(b,c) \,.\, \beta(a, b+c).$$

Soln : We know that

$$\beta(a,b) = \frac{\Gamma a \, \Gamma b}{\Gamma(a+b)}$$

L.H.S. of the problem $= \frac{\Gamma a \ \Gamma b}{\Gamma(a+b)} \cdot \frac{\Gamma(a+b) \ \Gamma c}{\Gamma(a+b+c)} = \frac{\Gamma a \, \Gamma b \, \Gamma c}{\Gamma(a+b+c)}$

R.H.S. of the problem $= \frac{\Gamma b \ \Gamma c}{\Gamma(b+c)} \cdot \frac{\Gamma a \ \Gamma(b+c)}{\Gamma(a+b+c)} = \frac{\Gamma a \ \Gamma b \ \Gamma c}{\Gamma(a+b+c)}$

Hence the result.

(4) Show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$$

and $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x .$$

Hence derive the expression for $J_{\pm 3/2}(x)$.

Soln : For the expression for $J_{1/2}(x)$ vide ex (2).

For $J_{-1/2}(x)$, put $n = -\frac{1}{2}$ in the expansion

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2.2\,(n+1)} + \frac{x^4}{2.4.2^2\,(n+1)\,(n+2)} \cdots\cdots\right]$$

we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} \cdots\cdots\right]$$

$$= \frac{x^{-1/2}}{2^{-1/2}\,\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cos x.$$

we know the recurrence relation

$$2\,n\,J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad \cdots\cdots\cdots\cdots (57)$$

Putting $n = 1/2$,

$$J_{1/2}(x) = x\left[J_{-1/2}(x) + J_{3/2}(x)\right]$$

or

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Again putting $n = -1/2$ in (57),

$$-J_{-1/2}(x) = x\left[J_{-3/2}(x) + J_{1/2}(x)\right]$$

or

$$J_{-3/2}(x) = -\frac{1}{x}J_{-1/2}(x) - J_{1/2}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

2.11. Summary of the lesson

Before going into the subject of Bessel function, it has been aimed at the basic requisites of beta and gamma functions which find their importance in this and future lesson. The series solution of Bessel equation is found. Generating function and the integral representation of $J_n(x)$ are given. Recurrence relations have been derived and examples are worked out. Orthogonal properties and Modified Bessel functions are given as special topics. A few miscellaneous examples are given.

2.12. Key terminology

Beta and Gamma function - Bessel differential equation - indicial equation - generating function - Modified Bessel function.

2.13 Self - Assessment questions

1. (a) Define beta and gamma functions and derive the relation connecting them.
   (b) Prove that $\beta(a, b+1) + \beta(a+1, b) = \beta(a, b)$.

2. (a) Show that $\Gamma\left(\frac{1}{2}-n\right) \cdot \Gamma\left(\frac{1}{2}+n\right) = (-1)^n \pi$.
   (b) Using beta and gamma functions, evaluate $\int_0^2 x\,(8-x^3)^{1/3}\,dx$.

3. (a) Prove that $J_{-n}(x) = (-1)^n J_n(x)$.
   (b) Show that $J_n(x)$ is even for even n and odd for odd n.

4. (a) Prove that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.
   (b) Evaluate $\int x^3 J_0(x)\,dx$.

5. (a) Find out $J_0(x)$ directly from the zeroth order Bessel differential equation.
   (b) Establish $x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x)$.

6. (a) Show that $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$.
   (b) Prove that $\int J_5\,dx = -\left[J_0 + 2 J_2 + 2 J_4\right]$.

2.14 Reference Books

1. B.S. Rajput "Mathematical Physics", Pragati rakashan, Meerut, 1999.
2. B.D. Gupta "Mathematical Physics", Vikas Publishing House, 1980.
3. K.V. Churchill "Operational Mathematics", Mc-Graw Hill Book Co., 1958.
4. E. Kreyszig "Advanced Engineering Mathematics", Wiley Eastern Pvt. Ltd., 1971.

Unit – I
Lesson – 3

HERMITE POLYNOMIALS

Objectives:

- To solve the Hermite differential equation in power series.
- To differentiate between Hermite polynomials and Hermite functions.
- To prove recurrence relations using generating function.
- To give Rodigue's formula and other differentiable forms.
- To give the integral representation of $H_n(x)$.

Structure:

3.1 Introduction:

Hermite polynomials are the power series solution of second order Hermite differential equation with variable coefficients. They will be need mainly as a mathematical tool in dromy of the scientific problem. Very familiar applications in quantum mechanics is the simple harmonic oscillator. While the ground state is given by the gaussiane function, higher states are given by products of the respective orders of Hermite polynomials with the gauseare functions.

3.2 Solution of Hermite's Differential Equation:

This equation is of the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2xy = 0 \quad \text{----------} \quad (1)$$

where y is a parameter.

Suppose its series solution is

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}. \quad a_0 \neq 0 \qquad \text{and k is a constant} \quad \text{----------} \quad (2)$$

So that $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k + r) x^{k+r-1}$

and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1) x^{k+r-2}$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get the identity

$$\sum_{r=0}^{\infty} [(k + r)(k + r - 1) x^{k+r-2} - 2(k + r - v) x^{k+r}] a_r \equiv 0 \quad \text{----------} \quad (3)$$

Equating the Coefficient of the first term (i.e. x^{k-2}) (by putting r = 0, to zero, we get

$a_0 k(k - 1) = 0$ giving k = 0, 1 as $a_0 \neq 0$ --------- (4)

Now, equating to zero the coefficient of second term (i.e. x^{k-1}) in (3) we get

$a_1 k (k + r) = 0$ i.e. $a_1 = 0$ when k = - 1 and a_1 may or may not be zero when k = 0, as the values of k are aheady fixed as in (4)

Also equating the coefficient of x^{k+r} to zero, we find

$a_{r+2} (k + r + 2)(k + r + 1) - 2a_r (k + r - v) = 0$

giving the recurrence relation

$$a_{r+2} = \frac{2(k + r - v)}{(k + r + 2)(k + r + 1)} a_r \quad \text{----------} \quad (5)$$

when k = 0, (5) becomes $a_{r+2} = \frac{2(r - v)}{(r + 2)(r + 1)} a_r$ ---------- (6)

and when k = 1, (5) becomes $a_{r+2} = \frac{2(1 + r - v)}{(r + 3)(r + 2)} a_r$ ---------- (7)

Case I. When k = 0, putting r = 0, 1, 2, 3,… in (6) we have

$$a_2 = -\frac{2}{\underline{|2}} v a_0; \quad a_3 = -\frac{2(v - 1)}{\underline{|3}} a_1$$

$a_4 = -\dfrac{2^2 v(v-2)}{\underline{|4}} a_0$; $a_5 = -\dfrac{2^2 (v-1)(v-3)}{\underline{|5}} a_1$ ………….. $a_{2r} \dfrac{(-2)^r v(v-2)...(v-2r+2)}{\underline{|2r}} a_0$;

$a_{2r+1} = \dfrac{(-2)^r (v-1)(v-3)...(v-2r+1)}{\underline{|2r+1}} a_1$

Now if $a_1 = 0$, then $a_3 = a_5 = a_7 = a_{2r+1} = \ldots = 0$.

But if $a_1 \neq 0$, then (2) gives for k = 0, $y = \sum_{r=1}^{\infty} a_r x^r$

i.e. $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$= a_0 + a_2x^2 + a_4x^4 + \ldots + a_1x + a_3x^3 + a_5x^5 + \ldots$

$$= a_0 \left[1 - \frac{2v}{\underline{|2}} x^2 + \frac{2^2 v(v-2)}{\underline{|4}} x^4 - \ldots + (-1)^r \frac{2^r}{\underline{|2r}} v(v-2)\ldots(v-2r+2)x^{2r} + \ldots\right]$$

$$+ a_1 x\left[1 - \frac{2(v-1)}{\underline{|3}} x^2 + \frac{2^2 (v-1)(v-3)}{\underline{|5}} x^4 - \ldots\right.$$

$$\left. + (-1)^r \frac{2^r}{\underline{|2r+1}} (v-1)(v-3)\ldots(v-2r+1)x^{2r} + \ldots\right] \quad \text{----------} \ (8)$$

$$= a_0 \left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{\underline{|2r+1}} v(v-2)\ldots(v-2r+2)x^{2r} + \ldots\right]$$

$$+ a_1 \left[x + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{2r+1} (v-1)(v-3)\ldots(v-2r+1)x^{2r+1} \ldots \right. \quad \text{----------} \ (9)$$

Case II: When k = 1, then $a_1 = 0$ and so by putting r = 0, 1, 2, 3, … in (7) we find

$a_2 = -\dfrac{2(v-1)}{\underline{|3}} a_0$

$a_4 = \dfrac{2^2 (v-1)(v-3)}{\underline{|5}} a_0$ …………….. $a_{2r} = (-1) \dfrac{2^r (2v-1)(v-3)...(v-2r+1)}{\underline{|2r+1}} a_0$

Hence the solution is

$$= a_0 x\left[1 - \frac{2(v-1)}{\underline{|3}} x^2 + \frac{2^2 (v-1)(v-3)}{\underline{|5}} x^4 - \ldots\right.$$

$$+\frac{(-1)^r\,2^r\,(v-1)(v-3)\ldots(v-2r+1)}{\underline{|2r+1}}x^{2r}+\ldots\Bigg] \quad \text{----------} \ (10)$$

clearly the solution (10) is included in the second part of (8) except that a_0 is replaced by a_1 and hence in order that the Hermite equation may have two independent solutions, a_1 must be zero, even if k = 0 and then (8) reduces to

$$y = a_0\left[1-\frac{2v}{\underline{|2}}x^2+\frac{2^2v(v-2)}{\underline{|4}}x^4-\ldots+(-1)^r\frac{2^r}{\underline{|2r}}v(v-2)\ldots(v-2r+2)x^{2r}+\ldots\right] \quad \text{----------} \ (11)$$

The complete integral of (1) is then given by

$$y = A\left[1-\frac{2v}{\underline{|2}}x^2+\frac{2^2v(v-2)}{\underline{|4}}x^4-\ldots\right]+Bx\left[1-\frac{2(v-1)}{\underline{|3}}x^2+\frac{2^2(v-1)(v-3)}{\underline{|5}}x^4-\ldots\right]$$

$$\text{----------} \ (12)$$

where A and B are arbitrary constants.

Where ***v*** is an integer, then the resulting solution is called Hermite Polynomial. The arbitrary constant A and B are taken as $(-1)^{v/2}.\frac{\underline{|v}}{\underline{\left|\frac{v}{2}\right.}}$ and $(-1)^{\frac{v-1}{2}}$ respectively.

In equation (12), the series with coefficient A alone is taken as the Hermite Polynomial of **<u>even</u>** order ***v*** and that with coefficient B alone is considered as Hermite Polynomial of **<u>odd</u>** order ***v***.

<u>3.3 Hermite Polynomials:</u>

The Hermite polynomial $H_n(x)$ is defined as

$$f(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{\underline{|n}} \quad \text{----------} \ (13)$$

for all integral values of n and all real values of x. (At a latter stage, it will be proved that the exponential function is the **<u>generating function of H_n (x))</u>** (13) can be written as

$$f(x,t) = e^{x^2}\,e^{-(x-t)^2} = \frac{H_0(x)}{\underline{|0}}+\frac{H_1(x)}{\underline{|1}}t+\frac{H_2(x)}{\underline{|2}}t^2+\ldots+\frac{H_n(x)}{\underline{|n}}t^n+\ldots$$

So that $$\left[\frac{\partial^n f(x,t)}{\partial t^n}\right]_{t=0} = \frac{H_n(x)}{\underline{|n}}\,\underline{|n} = H_n(x)$$

$$= \left[\frac{\partial^n}{\partial t^n}e^{-(x-t)^2}\right]_{t=0} e^{x^2} \quad \text{----------} \ (14)$$

If we put $x - t = p$ i.e. $t = x - p$ for $t = 0$ gives $x = p$

and $\frac{\partial}{\partial t} = -\frac{\partial}{\partial p}$ so that $\frac{\partial^n}{\partial t^n}\left\{e^{-(x-t)^2}\right\} = (-1)^n \frac{\partial^n}{\partial p^n} e^{-p^2}$

$$\therefore \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} = (-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2} = (-1)^n \frac{d^n}{dx^n} e^{-x^2} \quad \text{----------} \quad (15)$$

From (14) and (15), we therefore have

$$H_n(x) = e^{x^2} (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \text{(Rodrigue's formula)} \quad \text{----------} \quad (16)$$

From (16) we can calculate Hermite polynomials of various degrees such as

$$\left.\begin{array}{ll} H_0(x) = 1 & H_4(x) = 16x^4 - 48x^2 + 12 \\ H_1(x) = 2x & H_5(x) = 32x^5 - 160x^3 + 120x \\ H_2(x) = 4x^2 - 2 & H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120 \\ H_3(x) = 8x^3 - 12x & H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x \end{array}\right\} \quad \text{----------} \quad (17)$$

3.4 Recurrence formulae for $H_n(x)$ and to show that $H_n(x)$ is a solution of Hermite Equation:

Hermite equation is $y'' - 2xy' + 2ny = 0$ for integral values taking $\nu = n$.

$$\text{Also, } e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{\lfloor n} \quad \text{----------} \quad (18)$$

I Differentiating partially w.r.t. x, we have

$$2t\, e^{2tx-t^2} = \sum_{n=0}^{\infty} H'_n(x) \frac{t^n}{\lfloor n}$$

i.e $$2t \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{\lfloor n} = \sum_{n=0}^{\infty} H'_n(x) \frac{t^n}{\lfloor n}$$

which yields on equating the coefficients of $\frac{t^n}{\lfloor n}$ on either side,

$$2 \frac{\lfloor n}{\lfloor n-1} H_{n-1}(x) = H'_n(x)$$

i.e. $$2n\, H_{n-1}(x) = H'_n(x) \quad \text{-----------} \quad (19)$$

II Differentiating partially w.r.t. 't', both sides of (18) we get

$$2(x - t)\, e^{2tx-t^2} = \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{\lfloor n-1} \qquad \because n = 0 \text{ corresponds to the vanishing of R.H.S.}$$

or $2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{\lfloor n} - 2\sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{\lfloor n-1} = \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{\lfloor n-1}$

Equating the coefficients of t^n on either side we find

$$2x\frac{H_n(x)}{\lfloor n} - 2\frac{H_{n-1}(x)}{\lfloor n-1} = \frac{H_{n+1}(x)}{\lfloor n}$$

i.e. $2x\, H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$ ----------- (20)

III Eliminating $H_{n-1}(x)$ from (19) and (20) we get

$2x\, H_n(x) = H'_n(x) + H_{n+1}(x)$

or $H'_n(x) = 2x\, H_n(x) - H_{n+1}(x)$ ---------- (21)

IV Differentiating (21) w.r.t. x we find

$H_n''(x) = 2x\, H_n'(x) + 2\, H_n(x) - H'_{n+1}(x)$

Putting $H'_{n+1}(x) = 2(n+1)\, H_n(n)$ obtained from (19) on replacing n by n + 1; we have

$H_n''(x) = 2x\, H_n'(x) + 2\, H_n(x) - 2(n+1)\, H_n(x)$

or $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0$ ----------- (22)

which clearly follows that $y = H_n(x)$ as a solution of Hermite equation is proved by considering the generating function.

Q: Prove that

(i) $H_{2m}(0) = (-1)^m 2^{2m}\left(\frac{1}{2}\right)_m$

(ii) $H'_{2m+1}(0) = (-1)^m 2^{2m+1}\left(\frac{3}{2}\right)_m$

(iii) $H_{2m+1}(0) = 0$

(iv) $H'_{2m}(0) = 0$

(v) $\frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m \lfloor n}{\lfloor n-m} H_{n-m}(x)$, for m < n

<u>**Solution:**</u>

(i) Even Hermite polynomials are

$$H_{2m}(x) = \sum_{k=0}^{m} \frac{(-1)^k \lfloor 2m (2x)^{2m-2k}}{\lfloor k \lfloor 2m-2k}$$

$$\therefore\ H_{2m}(0) = \frac{(-1)^m \underline{|2m}}{\underline{|m}} = (-1)^m \frac{2m(2m-1)...3.2.1}{m(m-1)...3.2.1} = (-1)^m \frac{2^m \underline{|m}(2m-1)(2m-3)...3.1}{\underline{|m}}$$

$$= (-1)^m \frac{\underline{|m}}{\underline{|m}} .2^{2m}. \frac{2m-1}{2} . \frac{2m-3}{2} \ldots \frac{3}{2} . \frac{1}{2}$$

$$= (-1)^m\, 2^{2m} \frac{\underline{|m}}{\underline{|m}} \frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+m-1\right) = (-1)^m\, 2^{2m}\left(\frac{1}{2}\right)_m$$

(ii) From recurrence relation I, we have on replacing n by 2m + 1,

$H'_{2m+1}(x) = 2(2m+1)\, H_{2m}(x)$

$$\therefore\ H'_{2m+1}(0) = 2(2m+1)\, H_{2m}(0) = 2(2m+1).\ (-1)^m\, 2^{2m}\left(\frac{1}{2}\right)_m$$

by part (i)

$$= (2m+1).\ (-1)^m\, 2^{2m+1}\left[\frac{(2m-1)(2m-3)...3.1}{2^m}\right]$$

$$= (-1)^m\, 2^{2m+1}(2m+1)\left[\frac{3}{2}\cdot\left(\frac{3}{2}+1\right)\cdots\left(\frac{3}{2}+m-1\right)\right]$$

$$= (-1)^m\, 2^{2m+1}\left(\frac{3}{2}\right)_m$$

(iii) Odd Hermite Polynomials are

$$H_{2m+1}(x) = \sum_{k=0}^{2m+\frac{1}{2}} \frac{(-1)^k \underline{|2m+1}(2x)^{2m+1-2k}}{\underline{|k}(2m+1-2k)}$$

$\therefore\ H_{2m+1}(0) = 0$, since all terms containing x become zero.

(iv) From recurrence relation I, we have

$H'_{2m}(x) = 2(2m)\, H_{2m-1}(x)$

$\therefore\ H'_{2m}(0) = 4mH_{2m-1}(0)$

$= 0$ by (iii)

(v) From recurrence relation I, we have

$H'_n(x) = 2n\, H_{n-1}(x)$ ---------- (i)

i.e. $\dfrac{d}{dx}\{H_n(x)\} = 2n\, H_{n-1}(x)$

$$\therefore\ \frac{d^2}{dx^2}\{H_n(x)\} = 2n\frac{d}{dx}\{H_{n-1}(x)\} = 2n\ .\ 2(n-1)H_{n-2}(x)$$

by using (1)

$$= 2^2 n\,(n-1) H_{n-2}(x)$$

Similarly $\frac{d^3}{dx^3}\{H_n(x)\} = 2^3 n\,(n-1)(n-2) H_{n-3}(x)$

Proceeding similarly m times we find

$$\frac{d^m}{dx^m}\{H_n(x)\} = 2^m n\,(n-1)\ldots(n-m+2) H_{n-m}(x) \text{ where } m < n$$

$$= \frac{2^m \underline{|n}}{\underline{|n-m}} H_{n-m}(x)$$

3.5 Generating functions:

Q: To prove that $e^{2tx-t^2} = \sum_{r=0}^{\infty} \frac{t^n}{n!} H_n(x)$. Where e^{2tx-t^2} is called the generating function of $H_n(x)$.

Solution: We have

$$e^{2tx-t^2} = e^{2tx}\, e^{-t^2} = \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!} \sum_{r,s=0}^{\infty} \frac{(2x)^r}{r!s!} . t^{r+2s}$$

$\therefore$ Coefficient of t^n (for fixed value of s)

$$= (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!} \qquad \text{(put } r + 2s = n\text{)}$$

But the total coefficient of t^n is obtained by summing over all allowed values of s, (for $r = n - 2s \geq 0$)

$\therefore$ $n - 2s \geq 0$ i.e. $s \leq (n / 2)$. So we can say, that if n is even s goes from 0 to (n / 2) and if n is odd, s goes from 0 to (n – 2) / 2.

Hence required coefficient of $t^n = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!} = \frac{H_n(x)}{n!}$

(Here $\left[\frac{n}{2}\right]$ means the greatest integer that does not exceed $\frac{n}{2}$).

$$\Rightarrow e^{2tx-t^2} \sum_{n=1}^{\infty} \frac{t^n}{n!} H_n(x) \text{ i.e., } \quad e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H(x).$$

3.6 Another differential representation of Hermite polynomials ($H_n(x)$):

Q: Show that $H_n(x) = 2^n \left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n$

Solution: We have

$$\frac{1}{2}\frac{d}{dx}e^{(2tx)} = te^{2tx} \Rightarrow \frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right) = 2t^2 e^{2tx}$$

$$\therefore \frac{1}{2}\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right) = t^2 e^{2tx} \Rightarrow \left(\frac{1}{2}\frac{d}{dx}\right)^2 e^{2tx} = t^2 e^{2tx}$$

Continuing up to n times, we get

$$\left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx} = t^2 e^{2tx} \qquad \text{........(A)}$$

$$\Rightarrow \left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}. e^{2tx} = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n\right] e^{2tx}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n}. e^{2tx} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} t^{2n} e^{2tx} \qquad \text{(using A)}$$

$$= e^{2tx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} t^{2n} = e^{2tx}\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n$$

$$= e^{2tx} e^{-t^2} = e^{-t^2+2tx}$$

$$\Rightarrow \left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\sum_{n=0}^{\infty}\frac{1}{n!}(2tx)^n = \sum_{n=0}^{\infty}\frac{t^n}{n!} H_n(x)$$

[using generating function property]

Whence equating the coefficient of t^n from the two sides, we get

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\frac{1}{n!}2^n x^n = \frac{1}{n!} H_n(x) \text{ i.e., } H_n(x) = 2^n \left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n.$$

3.7 Hermite functions:

An equation closely related to Hermite equation is

$$\frac{d^2\psi}{dx^2} + (\lambda - x^2)\psi = 0 \quad \text{----------} \quad (23)$$

If we change the dependent variable ψ to y by the substitution

$$\psi = e^{-x^2/2}y \text{ ---------- } (24)$$

So that $\dfrac{d\psi}{dx} = e^{-x^2/2}\dfrac{dy}{dx} - y\ e^{-x^2/2}\cdot X$

and $\dfrac{d^2\psi}{dx^2} = e^{-x^2/2}\dfrac{d^2y}{dx^2} - 2x\,e^{-x^2/2}\dfrac{dy}{dx} - (e^{-x^2/2} - x^2 e^{-x^2/2})y$

we get from (1)

$$y'' - 2xy' + (\lambda - 1)y = 0 \text{ ----------- } (25)$$

If we put $\lambda - 1 = 2v$, then (25) reduces to Hermite equation i.e.

$$y'' - 2xy' + 2v\, y = 0$$

It therefore follows that the general solution of (23) is given by

$$\psi = e^{-x^2/2}y$$

where y is given by (12)

Thus if the parameter λ be of the form 1 + 2N, n being a positive integer, then the solution of (23) will be constant multiple of the function ψ_n defined by

$$\psi_n(x) = e^{-x^2/2}H_n(x) \text{ ---------- } (26)$$

where $H_n(x)$ is the Hermite polynomial of degree n.

Here the function $\psi_n(x)$ is said to be the Hermite function of order n.

Recurrence Relations for $\psi_n(x)$:

Differentiating (26) w.r.t. x, we have

$\psi'_n(x) = e^{-x^2/2}H'_n(x) - e^{-x^2/2}x\,H_n(x)$

$= 2n\ e^{-x^2/2}H_{n-1}(x) - x\,e^{-x^2/2}x\,H_n(x)$ $\quad\because\quad H'_n(x) = 2n\,H_{n-1}(x)$ by (19)

$= 2n\,\psi_{n-1}(x) - x\,\psi_n(x)$ using (26)

$\therefore 2n\,\psi_{n-1}(x) = \psi_n(x) + \psi'_n(x)$ ----------- (27)

Also from (25), $2x\,H_n(x) = 2n\,H_{n-1}(x) - H_{n+1}(x)$

Which may be expressed by using (26), as

$2x\,e^{-x^2/2}H_n(x) = 2n\,e^{-x^2/2}(x) + e^{-x^2/2}H_{n+1}(x)$

i.e., $2x\,\psi_n(x) = 2n\,\psi_{n-1}(x) + \psi_{n+1}(x)$ ---------- (28)

Eliminating $2n\,\psi_{n-1}(x)$ from (27) and (28) we find

$x\,\psi_n(x) + \psi'_n(x) = 2x\,\psi_n(x) - \psi_{n+1}(x)$

i.e., $\psi'_n(x) = x\,\psi_n(x) - \psi_{n+1}(x)$ ----------- (29)

3.8 Orthogonal Properties of Hermite polynomials:

Now since $H_n(x)$ is a solution of Hermite equation, we have

$H''_n(x) - 2x\, H'_n(x) + 2n\, H'_n(x) = 0$ by (22)

If we put $y = e^{-x^2/2}\, H_n(x)$ i.e., $H_n(x) = y\, e^{x^2/2}$

So that $H'_n(x) = y'\, e^{x^2/2} + xy\, e^{x^2/2}$

and $H''_n(x) = y''\, e^{x^2/2} + 2xy'\, e^{x^2/2} + y(1 + x^2)\, e^{x^2/2}$

then we get $y'' + (1 - x^2 + 2n)y = 0$ ---------- (30)

Since $y = e^{-x^2/2} H_n(x) = \psi_n(x)$ by (26), it therefore follows that $\psi_n(x)$ satisfies (30) and hence

$\psi''_n + (2n + 1 - x^2)\, \psi_n = 0$ ---------- (31)

for a function ψ_m, this relation is

$\psi''_m + (2m + 1 - x^2)\, \psi_m = 0$ ----------- (32)

Multiplying (31) by ψ_m; (32) by ψ_n and subtracting we get

$2(m - n)\, \psi_m\, \psi_n = \psi_m \psi''_n - \psi_n \psi''_m$ ---------- (33)

Integrating over $(-\infty, \infty)$, we have

$2(m - n)\int_{-\infty}^{\infty} \psi_m\, \psi_n\, dx = \int_{-\infty}^{\infty} (\psi_m \psi''_n - \psi_n \psi''_m)dx$

$= \left[\psi_m \psi'_n - \psi_n \psi'_m\right]_{-\infty}^{\infty} (\psi'_m\, \psi'_n - \psi'_n\, \psi'_m)dx$ (on integrating by parts)

$= 0$ $\because \psi_n(x) \to 0$ as $|x| \to \infty$ for all positive integral values of n.

or $\int_{-\infty}^{\infty} \psi_m\, \psi_n\, dx = 0$ if $m \neq n$

symbolically $I_{m,n} = \int_{-\infty}^{\infty} \psi_m\, \psi_n\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)dx = 0$

when $m \neq n$ ---------- (34)

In particular $I_{n-1,\, n+1} = 0$ ---------- (35)

Now from (28) we have $2x\, \psi_n(x) = 2n\psi_{n-1}(x) + \psi_{n+1}(x)$

$\therefore \int_{-\infty}^{\infty} 2x\, \psi_n(x)\, \psi_{n-1}\, dx = 2n \int_{-\infty}^{\infty} \psi_{n-1}(x) + \psi_{n+1}(x)$

$\because \int_{-\infty}^{\infty} \psi_{n-1} + \psi_{n+1}\, dx = 0$ by (35)

$= 2n\, I_{n-1,\, n+1}$ ---------- (36)

Also $\psi_n(x) = e^{-x^2/2}\, H_n(x)$

$= (-1)^n\, e^{x^2/2} \dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$ by (16)

Thus (36) gives

$$-\int_{-\infty}^{\infty} 2x\, e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)dx = 2n\, I_{n-1,\, n-1}$$

$$\text{or } 2n\, I_{n-1,\, n-1} = -\int_{-\infty}^{\infty} d\,(e^{x^2})\frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)dx$$

$$= \left[e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}).\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{x^2}$$

$$\left\{\frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)\right\}dx \qquad \text{(on integrating by parts)}$$

$$= 0 + I_{n,\, n} + I_{n+1,\, n-1}$$

$$= I_{n,\, n} \quad \text{by (35)}$$

$$\therefore I_{n,\, n} = 2n\, I_{n-1,\, n-1} \quad ---------- \quad (37)$$

Applying (37), repeatedly, we have

$$I_{n,\, n} = 2n\, I_{n-1,\, n-1} = 2n\, 2(n-1)\, I_{n-2,\, n-2}$$

$$= 2^2\, n(n-1).2(n-2)\, I_{n-3,\, n-3}$$

$$= 2^3\, n(n-1)\,(n-2)\, I_{n-3,\, n-3}$$

$$= \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$= 2^n\, n(n-1)\,(n-2)\,\ldots\ldots 3.2.1.\; I_{0,\, 0}$$

where $I_{0,\,0} = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ (From Beta and Gamma functions)

$$\therefore I_{n,\, n} = 2n\, \underline{|n}\, \sqrt{\pi} \quad ---------- \quad (38)$$

Combining the two results (34) and (38); we have in terms of Kronecker delta symbol

$$I_{m,\, n} = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)\, dx = 2^n\, \underline{|n}\, \sqrt{\pi}\, \delta_{m,\, n} \quad ---------- \quad (39)$$

Where $\delta_{m,\, n} = 0$ when $m \neq n$

$= 1$ when $m = n$.

(39) may also be written as

$$I_{m,\, n} = \int_{-\infty}^{\infty} \psi_m(x)\, \psi_n(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)\, dx$$

$$= 2^n\, \underline{|n}\, \sqrt{\pi}\, \delta_{m,\, n} \quad ---------- \quad (40)$$

Again $2x\, \psi_n(x) = 2n\, \psi_{n-1}(x) + \psi_{n+1}(x)$ gives

$$\int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx = nI_{m,\, n-1} + \frac{1}{2}\, I_{m,\, n+1}$$

$$= 0 \text{ for } m \neq n = 1$$

and $\int_{-\infty}^{\infty} x\, \psi_n \psi_{n+1}(x)dx = n\, I_{n+1,\, n-1} + \frac{1}{2}\, I_{n+1,\, n-1}$

$$= \frac{1}{2} 2^{n+1} \underline{|n+1} \sqrt{\pi} \text{ as above}$$

$$= 2^n \underline{|(n+1)} \sqrt{\pi} \quad \text{for } m = n$$

Hence $\int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx = 2^n \underline{|n+1} \sqrt{\pi}\, \delta_{m,\, n}$ ---------- (41)

Further $2n\, \psi_{n-1}(x) = x\, \psi_n(x) + \psi'_n(x)$ gives

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_m(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx$$

$$= 0 \text{ if } m \neq n = 1$$

and $2n\, I_{n-1,\, n-1} - 2^{n-1} \underline{|n} \sqrt{\pi}$ if $m = n = 1$

$$= 2^n \underline{|n} \sqrt{\pi} - 2^{n-1} \underline{|n} \sqrt{\pi} = 2^{n-1} \underline{|n} \sqrt{\pi}$$

Hence $\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2^{n-1} \underline{|n} \sqrt{\pi}\, \delta_{m,\, n}$ ---------- (42)

In the last if we take $m = n + 1$, then

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_{n+1}(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} \psi_{n+1}(x)\, \psi_n(x)\, dx$$

$$= -2^{n-1} \underline{|n} \sqrt{\pi}\, .$$

Q: Prove that $H_n(-x) = (-1)^n H_n(x)$

Solution: We have $\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{\underline{|n}} = e^{2tx - t^2} = e^{2tx}\, e^{t^2} = \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{\underline{|n}} \times \sum_{n=0}^{\infty} \frac{(-1) t^{2n}}{\underline{|n}}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k}}{\underline{|k}\, \underline{|n-2k}} t^n.$$

Equating coefficient of $\frac{t^n}{\underline{|n}}$ on either side, we get

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n} (2x)^{n-2k}}{\underline{|k}\, \underline{|n-2k}}$$

Replacing x by –x we get

$$H_n(-x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n} (-2x)^{n-2k}}{\underline{|k}\, \underline{|n-2k}}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (-1)^{n-2k} \underline{|n} (2x)^{n-2k}}{\underline{|k} \underline{|n-2k}}$$

$$= (-1)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n} (2x)^{n-2k}}{\underline{|k} \underline{|n-2k}}$$

$$= (-1)^n H_n(x)$$

Q: Prove $\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi}\, [2^{n-1} \underline{|n}\, \delta_{m, n-1} + 2^n \underline{|n+1}\, \delta_{n+1, m}]$

Solution: Integrating by parts we have

$$\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x)\, dx = \int -\frac{1}{2} e^{-x^2} H_n(x) H_m(x)\, dx \Big]_{-\infty}^{\infty}$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \frac{d}{dx} \{H_n(x) H_m(x)\} dx$$

$$= 0 + \int_{-\infty}^{\infty} e^{-x^2} \{H'_n(x) H_m(x) + H_n(x) H'_m(x)\} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} [2n H_{n-1}(x)\ H_m(x) + 2m H_n(x) H_{m-1}(x)\, dx] dx$$

by (20)

$$= n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) H_m(x)\, dx + \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_{m-1}(x)\, dx$$

$$= n \sqrt{\pi}\, 2^{n-1} \underline{|n-1}\, \delta_{m, n-1} + m \sqrt{\pi}\, 2^n \underline{|n}\, \delta_{n, m-1}$$

(by orthogonal properties)

$$= \sqrt{\pi}\, [2^{n-1} \underline{|n}\, \delta_{m, n-1} + 2^n \underline{|n+1}\, \delta_{n+1, m}]$$

$\because \delta_{n, m-1} = \delta_{n+1, m}$.

3.9 Integral representation of Hermite polynomials:

Let us find an equivalent expression for H_n in terms of a definite integral. If we put

$$y_n = \frac{1}{2\pi i} \oint z^{-n-1} e^{x^2 - (z-x)^2} dz \quad ---------- \quad (43)$$

and take the contour around a circle which has the origin as its center, then

$$\frac{dy_n}{dx} = \frac{1}{2\pi i} \oint 2z^{-n} e^{x^2 - (z-x)^2} dz \quad ---------- \quad (44)$$

and

$$\frac{d^2 y_n}{dx^2} = \frac{1}{2\pi i}\oint 4z^{-n+1} e^{x^2-(z-x)^2} dz$$

The differentiations here may be performed under the integral sign. When these derivatives are substituted on the left of the Hermite differential equation, it is found that

$$y'_n{}' - 2xy'_n + 2ny_n = \frac{1}{2\pi i}\oint (4z^2 - 4xz + 2n)\, e^{x^2-(z-x)^2} z^{-n-1} dz$$

$$= -\frac{1}{2\pi i}\oint \frac{d}{dz}(z^{-n} e^{x^2-(z-x)^2})dz = 0$$

The last step follows because the contents of the parenthesis, being a single-valued function of z, if n is an integer, takes the same value at the initial and final points of the contour integration. It has thus been shown that expression (43) is also a solution of Hermite's equation. Since it represents a polynomial in x it must be identical with H_n (x) except for a constant multiplier. So H_n (x) is proportional to $y_n(x)$ or $H_n(x) = k\, y_n(x)$, at x = 0,

$$H_n(x) = \frac{(-1)^{n/2} n!}{\left(\frac{n}{2}\right)!} \text{ for n even}$$

$$= 0 \text{ for n odd}$$

And from (43) $y_n(0) = \frac{1}{2\pi i}\oint z^{-n-1} e^{-z^2} dz = \frac{(-1)^{n/2}}{\underline{|n/2}}$ $\therefore k = \underline{|n}$

Hence $H_n(x) = \frac{\underline{|n}}{2\pi i}\oint z^{-n-1} e^{x^2-(z-x)^2} dz$ ---------- (45)

3.10 Summary:

The solution of Hermite differential equation is worked. Different series are obtained for even order polynomials and odd order polynomials. Hermite functions are defined as a product of Hermite polynomial with an exponential function. Generating function, recurrence relations, Rodrigue's formula, differential as well as integral representations and orthogonal properties are given with worked out samples.

3.11 Key Terminology:

Hermite Polynomials — Hermite functions — Generating function — Rodriguez formula — Orthogonal properties — Integral representation — Differential representations.

3.12 Self – assessment questions:

1. Solve the differential equation in power series

   $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2xy = 0$$

   when λ is an odd order integer. Hence obtain $H_3(x)$

2. Prove the Rodrigue's formula for Hermite polynomials

   $H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$ for all integral values of n. and hence find $H_3(x)$.

3. Following generating function of Hermite polynomial, hence that $y = H_n(x)$ is a solution of Hermite differential equation

   $y'' - 2xy' + 2xy = 0$

4. Show that $\frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m \underline{|n}}{\underline{|n-m}}.H_{n-m}(x)$ for $m<n$.

5. Prove that $\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n \underline{|n} \sqrt{\pi}\, \delta_{m,n}$

6. Show that $\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2^{n-1} \underline{|n} \sqrt{\pi}\, \delta_{m,n}$

   Where $\psi_n(x) = e^{-x^2/2} H_n(x)$ (See equation (42) of the lesson)

7. Prove that $H_n(-x) = (-1)^n . H_n(x)$.

8. Prove the recurrence relation of the Hermite function

   $2x\, \psi_n(x) = 2n\, \psi_{n-1}(x) + \psi_{n+1}(x)$.

3.13 Reference Books:

1. B.D. Gupta 'Mathematical Physics', Vikas publishing House, Sahibabad, 1980.

2. B.S. Rajput 'Mathematical Physics',Pragati Prakashan, 1999.

3. H. Margenau 'The Mathematics of Physics and Chemistry', Affiliated East – West Pvt. Ltd., 1971.

Unit – I
Lesson – 4

LAGUERRE POLYNOMIALS

Objectives:

- To find the solution of Laguerre differential equation.
- To derive associated Laguerre Polynomials.
- To provide integral representation of Laguerre Polynomials.
- To find the differential representation of Laguerre Polynomials.
- To prove the recurrence relations and orthonormalisation property of Laguerre Polynomials.

Structure:

4.1 Introduction:

There are very many particular differential equations which find all important place in the scientific applications. Laguerre's second order differential equation with variable coefficients is one such. Particular attention may be drawn to the radial wave equation in quantum mechanics isomorphous with the differential equation in mathematics whose solutions are associated Laguerre functions. But stress is given in this lesson to only Laguerre Polynomials.

4.2 Solution of Laguerre's Differential Equation: Laguerre's differential equation may be written as $xy'' + (1 - x) y' + \lambda y = 0$, where λ = constant ------------ (1)

This equation has a singularity at x = 0. But the singularity is non – essential or removable and hence the method of series integration is allowed by Fusch's theorem for solving this equation. For this purpose, we take$y = \sum_{l=0}^{\infty} a_l x^{K+l}$ (where k is constant and $a_0 \neq 0$) as the solution of given differential equation.

Thus $y' = \sum_l (k+l)\, a_l\, x^{k+l-1}$ and $y'' = \sum_l (k+l)(k+l-1)\, a_l\, x^{k+l-2}$

Substituting these values in equation (1), we have

$$\sum_l (k+l)^2 a_l x^{k+l-1} - \sum_l a_l (k+l-\lambda) x^{k+l} \equiv 0 \quad \text{---------} \quad (2)$$

This equation is true for all the values of x and hence the coefficients of all the powers of x are identically zero. As such equating to zero the coefficient of the lowest power of x, i.e., of x^{k-1}, we have the indicial equation as

$$k^2 a_0 = 0 \quad \text{----------} \quad (3)$$

since $a_0 \neq 0$**;** so equation (2) holds good only if k = 0. Then, we have

$$\sum l^2 a_l x^{l-1} - \sum a_l (l - \lambda) x^l = 0 \quad \text{---------} \quad (4)$$

Equating the coefficients of x^j to zero, we have $a_{j+1} = \dfrac{j - \lambda}{(j+1)^2} a_j$.

This is the recurrence relation for the coefficients.

Thus $\quad a_l = -\lambda a_0 = (-1)\lambda\, a_0,$

$$a_2 = \frac{1-\lambda}{2^2}(-\lambda a_0) = \frac{\lambda(\lambda-1)}{2^2} a_0 = \frac{\lambda(\lambda-1)}{(2!)^2} a_0 = (-1)^2 \frac{\lambda(\lambda-1)}{(2!)^2} a_0.$$

$$a_3 = \frac{2-\lambda}{3^2} a_0 = \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0 = (-1)^3 \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_r = \frac{r - 1 - \lambda}{r^2} a_{r-1} = (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r!)^2} a_0.$$

$$\text{So } y = \sum_{l=0}^{\infty} a_l x^l = a_0 \left[1 - \lambda x + \frac{\lambda(\lambda-1)}{(2!)^2} x^2 - \cdots + (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r!)^2} x^r + \ldots \right] \text{-------} (5)$$

If λ = n, a positive integer, and if, we put $a_0 = n\,!$, (some authors may take $a_0 = 1$) then the solution for y contains only (n + 1) terms and becomes the Laguerre Polynomial of degree n.

Thus (5) becomes $L_n(x) = \displaystyle\sum_{r=0}^{n} \frac{(-1)^2 !(\underline{|n})^2}{\underline{|n - r}(\underline{|r})^2} x^r$

$$= (-1)^2 \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2 (n-1)^2}{2!} x^{n-2} + \cdots + (-1)^n n! \right] \quad \text{----------} \quad (6)$$

This is the expression for **Laguerre's Polynomial.**

Equation (6) gives $L_n(0) = n!$, $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = x^2 - 4x + 2$,

$L_3(x) = x^3 + 9x^2 - 36x + 6$, $L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 48$, and so on.

Thus a Laguerre's polynomial is the solution of equation

$$xL_n''(x) + (1 - x)L_n'(x) + nL_n(x) = 0 \quad \text{----------} \quad (7)$$

4.3 Associated Laguerre's Polynomials:

Differentiating equation (7) p times, we have

Ordinary Differential Equations and Useful Polynomials

$$x \frac{d^{p+2}L}{dx^{p+2}} + (p+1-x) \frac{d^{p+1}L}{dx^{p+2}} + (n-p) \frac{d^pL}{dx^p} = 0 \quad \text{----------} \quad (8)$$

If we substitute $y = \dfrac{d^pL_n(x)}{dx^p}$ in this equation then we get

$xy'' + (p + 1 - x)y' + (n - p)y = 0.$ where p is an integer ≥ 0 ---------- (9)

Thus the solution of equation (9) is

$$y = \frac{d^pL_n(x)}{dx^p} = L_n^p(x). \quad \text{----------} \quad (9a)$$

This is called the associated Laguerre polynomial of degree $n - p$.

Let v satisfying associated Laguerre differential equation

$$xv'' + (p + 1 - x)v' + (n - p)v = 0 \quad \text{----------} \quad (10)$$

be related with another function y by the relation

$$y = e^{-x/2} x^{(p-1)/2} v$$

or

$$v = ye^{-x/2} x^{(p-1)/2}.$$

Substituting this expression of v in equation (10), we get

$$xy'' + 2y' + \left[n - \frac{p-1}{2} - \frac{4}{x} - \frac{p^2-1}{4x} \right] y = 0 \quad \text{----------} \quad (11)$$

As seen from equation (9a), $v = L_np(x)$.

So solution of equation (11) is $y = e^{-x/2} x^{(p-1)/2} L_np(x) = y_n, p$ ---------- (12)

This function y is called an **Associated Laguerre Function.**

4.4 Integral representation of Laguerre Polynomials:

Let us assume the integral

$$y_n = \frac{1}{2\pi i}\oint \frac{\rho^{-n-1}}{1-\rho} e^{-x\rho(1-\rho)}\, d\rho \quad \text{such that} \quad y_n' = \frac{1}{2\pi i}\oint \frac{\rho^{-n}}{(1-\rho)^2} e^{-x\rho(1-\rho)}\, d\rho$$

$$\text{and } y_n'' = \frac{1}{2\pi i}\oint \frac{\rho^{-n+1}}{(1-\rho)^3} e^{-x\rho(1-\rho)}\, d\rho \quad \text{----------} \quad (13)$$

Laguerre's polynomial is the solution of the differential equation

$xy'' + (1 - x)\, y' + ny = 0$ ---------- (14)

If we substitute in this equation the values of y_n, y_n and y_n'' as given by equation (13), then we have

$$\frac{1}{2\pi i}\oint \left[\frac{x\rho^2}{(1-\rho)^2} - \frac{(1-x)\rho}{1-\rho} + n\right] \frac{\rho^{-n-1}}{1-\rho} e^{-x\rho(1-\rho)}\, d\rho$$

for the left hand side of equation (14). It may also be written as

$$-\frac{1}{2\pi i}\oint \frac{d}{d\rho}\left[\frac{\rho^{-n}}{1-\rho} e^{-x\rho(1-\rho)}\right] d\rho \quad ,$$

Which is equal to zero since quantity in the bracket takes same values at the initial and final points of the closed contour.

So L.H.S. of equation (14) is zero if $y = \frac{1}{2\pi i}\oint \frac{\rho^{-n-1}}{1-\rho} e^{-x\rho(1-\rho)}\, d\rho$, and hence this value of y represents a solution of the Laguerre's equation and hence we may have

$L_n(x) = cy_n, (x)$,

Where c is a constant. If, we substitute x = 0 in equation (6), then, we have

$$L_n(0) = n! \quad \text{----------} \quad (15)$$

Substituting x = 0 in the first equation of (13), we get $y_n(0) = \frac{1}{2\pi i}\oint \frac{\rho^{-n-1}}{1-\rho} d\rho$

But the integral $\oint \frac{\rho^{-n-1}}{1-\rho} d\rho$ may be calculated to be $2\pi i$ by the method of contour integration where the contour includes origin. In this way, we get

$$y_n(0) = 0 \quad \text{-----------} \quad (16)$$

By comparing equations (15) and (16), we get c = n ! or $L_n(0) = n!\, y_n(0)$.

So $\quad L_n = n!\, y_n$

But $y_n = \dfrac{1}{2\pi i}\oint \dfrac{\rho^{-n-1}}{1-\rho} e^{-x\rho(1-\rho)} d\rho$

Thus $L_n = \dfrac{n!}{2\pi i}\oint \dfrac{\rho^{-n-1}}{1-\rho} e^{-x\rho(1-\rho)} d\rho$ ---------- (17)

But y_n given by the first of equation (13) when calculated by contour integration is found equal to the coefficient of ρ^n in the expression $(1-\rho)^{-1}e^{-x\rho/(1-\rho)}$.

Actually, y_n should be the coefficient of $\dfrac{1}{\rho}$ or ρ^{-1} in the Laurent equation of $\dfrac{\rho^{-n-1}}{1-\rho} e^{\frac{-x\rho}{1-\rho}}$ i.e.,

the coefficient of ρ^n in the expansion of $(1-\rho)^{-1} e^{\frac{-x\rho}{1-\rho}}$.

So, we have $(1-\rho)^{-1} e^{\frac{-x\rho}{1-\rho}} = \sum_{n=0}^{\infty} y_n \rho^n = \sum_{n=0}^{\infty} \dfrac{L_n(x)}{n!}\rho^n$ ---------- (18)

This is generating function for Laguerre's polynomial.

4.5 Recurrence formulae for Laguerre Polynomials:

(1) L_n is solution of equation $xy'' + (1-x)y' + ny = 0$

So, we have $xL_n'' + (1-x)L_n' + nL_n = 0$ ---------- (19)

(ii) Differentiating equation (18) with respect to ρ, we have

$$\frac{1-x-\rho}{(1-\rho)^3} e^{-x\rho/(1-\rho)} \sum_{n=1}^{\infty} \frac{L_n(x)\rho^{n-1}}{(n-1)!} = \sum_{\lambda=1}^{\infty} \frac{L_\lambda(x)\rho^{\lambda-1}}{(\lambda-1)!}$$

or $$(1-x-\rho)\frac{e^{-x\rho/(1-\rho)}}{(1-\rho)^3} = (1-\rho)^2 \sum_{\lambda=1}^{\infty} \frac{L_\lambda(x)\rho^{\lambda-1}}{(\lambda-1)!}$$

or $$(1-x-\rho)\sum_{\lambda=0}^{\infty} \frac{L_\lambda(x)\rho^{\lambda}}{(\lambda-1)!} = (1-2\rho+\rho^2)\sum_{\lambda=1}^{\infty} \frac{L_\lambda(x)\rho^{\lambda-1}}{(\lambda-1)!}$$

Equating the coefficients of ρ^n on both the sides of this equation, we have

$$(1-x)\frac{L_n}{n!} - \frac{L_{n-1}}{(n-1)!} = \frac{L_{n+1}}{n!} - \frac{2L_n}{(n-1)!} + \frac{L_{n-1}}{(n-2)!}$$

or $(1 + 2n - x) L_n - n^2 L_{n-1} - L_{n+1} = 0$ ---------- (20)

(iii) Differential generating function equation (18) w.r.t. x, we have

$$-(1-\rho)^{-1}\frac{\rho}{1-\rho} e^{-x\rho/(1-\rho)} = \sum_{n=0}^{\infty} \frac{L'_n(x)}{n!}\rho^n$$

or $-\rho\,(1-\rho)^{-1}\, e^{-x\rho/(1-\rho)} = (1-\rho) \sum_{n=0}^{\infty} \frac{L'_n(x)}{n!} \rho^n$

or $-\Sigma\rho\, \frac{L_n(x)}{n!} \rho^n = (1-\rho) \sum_{n=0}^{\infty} \frac{L'_n(x)}{n!} \rho^n$

Equating the coefficients of ρ^n on both the sides of this equation, we get

$$\frac{L_{n-1}(x)}{(n-1)!} = \frac{L_n'(x)}{n!} - \frac{L'_{n-1}(x)}{(n-1)!}$$

$$L_n'(x) = n\,L'_{n-1}(x) - nL_{n-1}(x) \quad \text{------------} \quad (21)$$

4.6 Differential Formula for Laguerre's polynomial (Rodrigue Formula):

Differentiating generating function equation (18) n times w.r.t. ρ, we have

$$e^x \frac{\partial^n}{\partial\rho^n}\left[(1-\rho)^{-1} e^{-x/(1-\rho)}\right] = L_n(x) + L_{n+1}(x)\,\rho + \ldots \quad \text{----------} \quad (22)$$

since all terms up to the term containing ρ^{n-1} vanish when differentiated n times.

But $\frac{\partial}{\partial\rho}\left[(1-\rho)^{-1} e^{-x/(1-\rho)}\right] = \frac{1-x-\rho}{(1-\rho)^3}\, e^{-x/(1-\rho)}$

So $\lim_{\rho\to 0} \frac{\partial}{\partial\rho}\left[(1-\rho)^{-1} e^{-x/(1-\rho)}\right] = (1-x)e^{-x} = \frac{d}{dx}(x\, e^{-x})$.

Similarly, $\lim_{\rho\to 0} \frac{\partial^2}{\partial\rho^2}\left[(1-\rho)^{-1} e^{-x/(1-\rho)}\right] = \frac{d^2}{dx^2}(x^2 e^{-x})$ and so on.

Thus finally, we have $\lim_{\rho\to 0} \frac{\partial^n}{\partial\rho^n}\left[(1-\rho)^{-1} e^{-x/(1-\rho)}\right] = \frac{d^n}{dx^n}(x^n e^{-x})$

And hence equation (22) for $\rho \to 0$ gives

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \quad \text{----------} \quad (23)$$

Which is Rodrigue's representation of Laguerre's polynomial.

The Rodrigue's; representation of associated Laguerre's polynomial is given by

$$L_n^k(x) = e^x x^{-k} \frac{d^n}{dx^n}(e^{-x} x^{n+k}) \quad \text{----------} \quad (24)$$

4.7 Orthogonality of Laguerre Polynomials:

Laguerre's differential equation is not self – adjoint and thus Laguerre's polynomials $L_n(x)$ do not by themselves form an orthogonal set.

However, the related set of functions

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x) \quad ---------- \quad (25)$$

where $e^{-x/2}$ is the weight function of $L_n(x)$ is orthogonal for the interval $0 \le x \le \infty$, i.e.,

$$\int_0^\infty e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} dx = \int_0^\infty \phi_m(x)\, \phi_n(x)\, dx = \delta_{m,n} \quad ---------- \quad (26)$$

It can be proved as follows:

We know that $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$. Multiplying both sides with $e^{-x} x^m$ and integrating w.r.t. x between the limits 0 to ∞, we get

$$\int_0^\infty e^{-x} x^m L_n(x)\, dx = \int_0^\infty x^m \frac{d^n}{dx^n}(x^n e^{-x}) dx$$

$$= \left[x^m \frac{d^{n-1}}{dx^{n-1}} x^n e^{-x} \right]_0^\infty - \int_0^\infty m x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx = (-1) m \int_0^\infty x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx$$

$$= (-1)^2 m(m-1) \int_0^\infty x^{m-2} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x}) dx = \ldots\ldots\ldots\ldots$$

$$= (-1)^2 \,.\, m! \int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x}) dx \qquad \text{(on integrating by parts)}$$

$= 0$ if $n > m$.

similarly, $\int_0^\infty e^{-x} x^m L_m(x)\, dx = 0$ if $m > n$.

But $L_n(x)$ is a polynomial of degree n in x and $L_m(x)$ is a polynomial of degree m in x.

Therefore, $\int_{-\infty}^\infty e^{-x} L_m(x) L_n(x)\, dx = 0$ for $m > n$ and for $m < n$

Or $$\int_{-\infty}^\infty e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} dx = 0 \text{ if } m \ne 0. \quad ---------- \quad (27)$$

For m = n, $\int_{-\infty}^\infty e^{-x} \{L_n(x)\}^2 dx = (-1)^n \int_{-\infty}^\infty e^{-x} x^n L_n(x)\, dx$

(since the term of degree n in $L_n(x)$ is $(-1)^n x^n$).

Thus $\int_0^\infty e^{-x} \{L_n(x)\}^2 dx = (-1)^n \int_0^\infty e^{-x} x^n e^x \frac{d^n}{dx^n}(x^n e^{-x}) dx$

$$= (-1)^n n! \int_0^\infty x^n (-1)^n e^{-x} dx = n! \int_0^\infty x^n e^{-x} dx = (n!)^2 \quad --------- \quad (28)$$

or $$\int_0^\infty \frac{1}{n!} e^{-x/2} L_n(x) \cdot \frac{e^{-x/2}}{n!} L_n(x)\, dx = 1$$

Thus from equations (27) and (28), we get

$$\int_0^\infty e^{-x/2} \frac{L_m(x)}{m!} e^{-x/2} \frac{L_n(x)}{n!} dx = \delta_{m,n}$$

$$\int_0^\infty \phi_m(x)\, \phi_n(x)\, dx = \delta_{m,n}$$

Second method: To prove that $\int_0^\infty e^{-x} \frac{L_m(x)}{\underline{|m}} \cdot \frac{L_n(x)}{\underline{|n}}\, dx = \delta_{m\,n}$

Proof: We have $\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{\underline{|n}} = \frac{1}{1-t} e^{\frac{-tx}{1-t}}$

and $\sum_{m=0}^{\infty} s^m \frac{L_m(x)}{\underline{|m}} = \frac{1}{1-s} e^{\frac{-sx}{1-s}}$

Further,

$$\sum_{m,n=0}^{\infty} e^{-x} t^n s^m \frac{L_n(x)}{\underline{|n}} \frac{L_m(x)}{\underline{|m}} = e^{-x} \frac{1}{1-t} \frac{1}{1-s} e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}}$$

Integrating both sides w.r.t. x between the limits 0 to ∞, we can have a typical integral

$\int_0^\infty e^{-x} \frac{L_n(x)}{\underline{|n}} \cdot \frac{L_m(x)}{\underline{|m}}\, dx$ = coefficient of $t^n s^m$ in the expansion of

$$\int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)} e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}} \cdot dx$$

But $\int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)} e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}} = \frac{1}{(1-t)(1-s)} \int_0^\infty e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \cdot dx$

$$= \frac{1}{(1-t)(1-s)\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \cdot \left[e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]}\right]_{\lambda=0}^{\infty}$$

$$= \frac{1}{-1+ts}(0-1) = \frac{1}{1+ts} = [1 + ts + (ts)^2 + \ldots\ldots.]$$

Here the coefficient of $t^n s^m$ is zero ($m \neq n$) and 1 for $m = n$.

$$\therefore \int_0^\infty e^{-x} \frac{L_n(x)}{\lfloor n} \cdot \frac{L_m(x)}{\lfloor m} \, dx = \delta_{mn}$$

4.8 Examples:

1. Show that the generating function

$$\frac{1}{1-t} e^{\frac{-tx}{1-t}} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} L_n(x).$$

Solution: We have

$$\frac{1}{1-t} e^{\frac{-tx}{1-t}} = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{1}{\lfloor r} \cdot \left(-\frac{xt}{1-t}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} \cdot \frac{x^r t^r}{(1-t)^{r+1}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} \cdot x^r t^r (1-t)^{-(r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} x^r t^r \cdot \left[1 + (r+1)t + \frac{(r+1)(r+2)}{\lfloor 2} t^2 + \cdots\cdots\right]$$

coefficient of $t^n = \frac{(-1)^n}{\lfloor n} x^n + \frac{(-1)^{n-1}}{\lfloor n-1} \cdot nx^{n-1} + \frac{(-1)^{n-2}}{\lfloor n-2} \cdot n(n-1)x^{n-2} + \cdots\cdots$

$$= \frac{(-1)^n}{\lfloor n}\left[x^n - n^2 x^{n-1} + \frac{n^2(n-1)^2}{\lfloor 2} x^{n-2} + \cdots\cdots + (-1)^n \cdot \frac{(\lfloor n)^2}{\lfloor n}\right]$$

$$= \frac{L_n(x)}{\lfloor n} \qquad \text{as per equation (6)}$$

$$\therefore \frac{1}{1-t} e^{\frac{-tx}{1-t}} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} L_n(x).$$

2. Prove that $x L'_n = n L_n - n^2 L_{n-1}$

Solution: We know the recurrence relations

$(1 + 2n - x) L_n - n^2 L_{n-1} - L_{n+1} = 0$ ---------- (20)

and $L'_n = n L'_{n-1} - n L_{n-1}$ ---------- (21)

Differentiating (20) w.r.t. x, we get

$(1 + 2n - x) L'_n - L_n - n^2 L'_{n-1} - L'_{n+1} = 0$

or $(1 + 2n - x) L'_n - L_n - n^2 L'_{n-1} - (n + 1)L'_n + (n+1)L_n = 0$ from (21)

(ie) $(n - x) L'_n - n^2 L'_{n-1} + n L_n = 0$

or $(n - x) L'_n - n(L'_n + n L_{n-1}) + n L_n = 0$ from (21)

(ie) $x L'_n = n L_n - n^2 L'_{n-1}$

3. Show that $L'_n(x) = -\underline{|n} \sum_{r=0}^{n-1} \frac{L_r(x)}{\underline{|r}}$

Solution: We know that $\sum_{n=0}^{\infty} \frac{t^n}{\underline{|n}} L'_n(x) = \frac{1}{1-t} e^{\frac{-tx}{1-t}}$

Differentiating w.r.t. x,

$$\sum_{n=0}^{\infty} \frac{t^n}{\underline{|n}} L'_n(x) = -\frac{t}{1-t} \cdot \left(\frac{1}{1-t} . e^{\frac{-tx}{1-t}}\right)$$

$$= -t (1-t)^{-1} \sum_{r=0}^{\infty} \frac{t^r}{\underline{|r}} L_r(x).$$

$$= -t (1 + t + t^2 + \ldots..) \sum_{r=0}^{\infty} \frac{t^r}{\underline{|r}} L_r(x).$$

or equating the coefficient of t^n on both sides,

$$\frac{1}{\underline{|n}} L'_n(x) = -1 \cdot \frac{L_{n-1}(x)}{\underline{|n-1}} - 1.\frac{L_{n-2}(x)}{\underline{|n-2}} \ldots\ldots - 1\frac{L_0(x)}{\underline{|0}}$$

$$= -\sum_{r=0}^{n-1} \frac{L_r(x)}{\underline{|r}}$$

(ie) $L'_n(x) = -\underline{|n} \sum_{r=0}^{n-1} \frac{L_r(x)}{\underline{|r}}$

4. If a function f(x) defined in $(0, \infty)$ is expressed as $f(x) = C_0 L_0(x) + C_1 L_1(x) + C_2 L_2(x) + \ldots\ldots$ then show that

$$C_k = \frac{\int_0^{\infty} e^{-x} L_k(x) f(x)\, dx}{\int_0^{\infty} e^{-x} [L_k(x)]^2\, dx} \qquad k = 0, 1, \ldots..$$

Solution: Multiply f(x) $C_0\, L_0(x) + C_1\, L_1(x) + C_2\, L_2(x) + \ldots\ldots$ with $e^{-x}\, L_k(x)$ and integrate w.r.t. x, between the limits 0 to ∞, we get

$$\int_0^\infty e^{-x}\, L_k(x)\, f(x)dx = C_0 \int_0^\infty e^{-x}\, L_k(x)\, L_0(x)\, dx + C_1 \int_0^\infty e^{-x}\, L_k(x)\, L_1(x)\, dx + \ldots.$$

$$+ C_k \int_0^\infty e^{-x}\, L_k(x)\, L_k(x)\, dx + \ldots\ldots$$

By the orthogonal property of Laguerre polynomial that

$$\int_0^\infty e^{-x}\, L_m(x)\, L_n(x)\, dx = C_k \int_0^\infty e^{-x}\, [L_k(x)]^2\, dx$$

or $$C_k = \frac{\int_0^\infty e^{-x} L_k(x) f(x)\, dx}{\int_0^\infty e^{-x} [L_k(x)]^2\, dx}.$$

5. Find the value of $L_2(x)$ and evaluate $\int_0^\infty e^{-x}\, L_2(x)\, x^m\, dx$ where m is a +ve integer.

Solution: We know the Laguerre polynomial of degree n as

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2}{\underline{|1}} k^{n-1} + \frac{n^2 (n-1)^2}{\underline{|2}} x^{n-2} + \cdots\cdots + (-1)^n . \underline{|n} \right]$$

$$\therefore\ L_2(x) = (-1)^2 \left[x^2 - 4x + \frac{2^2 . 1^2}{\underline{|2}} \right]$$

$$= x^2 - 4x + 2.$$

Now $$\int_0^\infty e^{-x}\, L_2(x)\, x^m\, dx = \int_0^\infty e^{-x}\, (x^2 - 4x + 2)\, x^m\, dx$$

$$= \int_0^\infty e^{-x}\, x^{m+2}\, dx - 4 \int_0^\infty e^{-x}\, x^{m+1} .\, dx + 2 \int_0^\infty e^{-x}\, x^m\, dx.$$

$$= \quad \sqrt{m+3} - 4\sqrt{m+2} + 2\sqrt{m+1}$$

4.9 Summary:

Similar to the previous lessons, the structure of this lesson is also the same and it started with the second order Laguerre differential equation. Having obtained the solution, one finds that choice of the arbitrary constant $\mathbf{a_0}$ is of two ways. So while dealing with these polynomials caution should be exercised in asking and answering generations. No doubt, associated Laguerre polynomials are more useful. However, due to their unwieldy nature, they are only introduced and stress is given to Laguerre polynomials.

Integral and differential representations of Laguerre polynomials are given. Using the generating function, recurrence relations have been proved. Orthonormalization property, using weight function, is proved. Typical examples are worked out and probable questions are given.

4.10 Key Terminology:

Laguerre differential equation — Associated Laguerre polynomials — Generating function — Rodrigue's Formula — Integral representations — Weight functions — Recurrence relations.

4.11 Self – assessment questions:

1 Obtain the generating function of Laguerre Polynomial from its integral representation.

2 State and prove the Orthonormalization property of Laguerre Polynomials.

Or

Show that $\phi_n(x) = e^{-x/2} L_n(x)$ from an orthonormal set.

3. If $L_n(x)$ is the Laguerre Polynomial of order n show that $L_n(x) = e^x \dfrac{d^n(x^n e^{-x})}{dx^n}$.

4. Prove the recurrence relation for Laguerre Polynomial

$$(1 + 2n - x) L_n(x) - n^2 L_{n-1}(x) - L_{n+1}(x) = 0$$

5. Show that $L_n^m(x)$ is a solution of $xy'' + (m + 1 - x) y' + (n - m) y = 0$ when m is an integer $m \geq 0$

6. Find the expression of $L_4(x)$ and show that $L_4^2(x) = 144 - 96x + 12x^2$.

4.12 Reference Books:

1. B.S. Rajput 'Mathematical Physics' Pragati Prakashan, Meerut,1999.

2. H. Margenau and G.M. Murphy 'The Mathematics of Physics and Chemistry' Affiliated East – West Press Pvt. Ltd., 1971.

3. P.P. Gupta, R.P.S. Yadav and G.S. Malik, 'Mathematical Physics', Kedarnath Ramnath, Meerut, 1980.

Unit – II
Lesson – 5

ANALYTIC FUNCTIONS

Objectives:

- To introduce the basic concepts of complex numbers and functions.
- To give the definitions of basic parameters and terminology in the complex region.
- To understand the concept of many-valued and single valued functions.
- To give the definition of analytic functions and to derive Cauchy – Riemann equations.
- To bring the relation between analytic functions and harmonic functions.

Structure:

5.1 Introduction:

Many scientific problems may be treated and solved by methods of complex analysis. These problems can be subdivided into two large classes. The first class consists of elementary problems dealing with electric circuits, vibrating systems etc., for which the knowledge of complex numbers gained in college Algebra and calculus is sufficient. The second class of problems such as theory of heat, fluid dynamics etc., requires a detailed knowledge of the theory of complex analytic functions.

It will be seen that the real and imaginary parts of an analytic function are solutions of Laplace's equation in two independent variables. Consequently, two dimensional problems can be treated by methods developed in connection with analytic functions. There is, however, large area of applications in scientific problems in which familiarity with the theory of complex functions beyond this minimum is indispensable.

5.2 Basic concepts:

We consider a complex number as having the form a + ib where a and b are real numbers and i, which is called the imaginary number, has the property that $i^2 = -1$. If z = a + ib, then a is called the real part of z and b is called the imaginary part of z and are denoted by Re (z) and Im (z) respectively. The symbol z stands for complex variable. The complex conjugate or simply conjugate of z often denoted by $\bar{z}$ or z^* is given by a – ib. The absolute value or modulus of a complex number or briefly mod z or |z| is given by $|z| = |a + ib| = \sqrt{a^2 + b^2} = |\bar{z}|$. Further $z\bar{z} = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2$ which is an important property.

Since a complex number x + iy can be considered as an ordered pair of real numbers (x, y), we can represent complex numbers by means of the representative points (x, y) in two dimensional xy – plane called Argand plane in which x – axis is taken as real axis and y – axis as imaginary axis as shown in the figure 1.

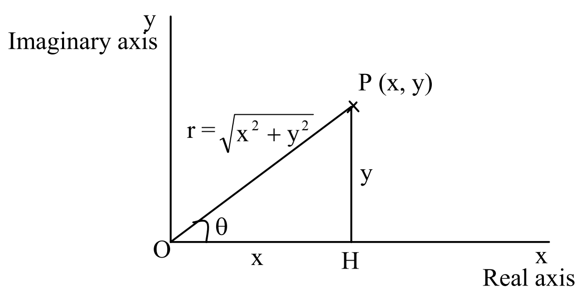


Fig 1: Argand diagram.

Further if (r, θ) are the polar coordinates, then x = r cosθ and y = r sinθ. So the complex number can also be represented as $z = x + iy = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$ by Euler's formula.

Consider $z_1 = x_1 + iy_1 = r_1(\cos\theta + i\sin\theta) = r_1 e^{i\theta 1}$

where $r_1 = |z_1| = \sqrt{x_1^2 + y_1^2}$ and $\theta_1 = \text{amp } z_1 = \tan^{-1}\dfrac{y_1}{x_1}$ is called the amplitude of z_1 or argument of z_1 (arg z_1).

Similarly consider $z_2 = x_2 + iy_2 = r_2 e^{i\theta 2}$, Then

$z_1 z_2 = r_1 r_2 e^{i(\theta 1 + \theta 2)}$ in which

$| z_1 z_2 | = r_1 r_2 = | z_1| |z_2 |$ and

amp $(z_1 z_2)$ = amp z_1 + amp z_2.

(i.e) modulus of a product of complex numbers is equal to product of the moduli of the individual complex numbers. And amplitude of the product of complex numbers is the sum of the amplitudes of individual complex numbers.

A number ω is called as **nth root** of a complex number z if we write

$$\omega = z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \qquad \text{for } k = 0,1,\ldots\ldots.n-1$$

In particular, if $z = 1 = 1.e^{i0}$, then

$$\omega = 1^{1/n} = \cos\left(\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$$

$$= 1,\ e^{\frac{2\pi}{n}i},\ e^{\frac{4\pi}{n}i},\ldots\ldots. \text{ are the nth roots of unity.}$$

In w = f(z), if to each value of z, there corresponds only one value to w, then w is called a **single** valued function of z

Example: If $w = z^2$, then for a single value z = 4 there corresponds one value to w as $4^2 = 16$.

So $w = z^2$ is single valued. On the other hand if $w = z^{\frac{1}{2}}$, then for a single value of z = 4, there corresponds two values to w as + 2 and –2. Thus its is a double valued or generally called as **many valued function.**

Q: Show that the modulus of the sum of two complex numbers does never exceed the sum of their moduli.

Solution: Let z_1 and z_2 be the two complex numbers and their conjugates are $\overline{z}_1$ and $\overline{z}_2$

Now $| z_1 + z_2 |^2 = (z_1 + z_2) \left(\overline{z_1 + z_2}\right) = (z_1 + z_2) \left(\overline{z}_1 + \overline{z}_2\right)$ $\qquad (\because z^2 = z\overline{z})$

$$= z_1\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_2 + z_2\overline{z}_1$$

$$= |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z_1\overline{z}_2}$$

$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\overline{z}_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2\,| z_1\overline{z}_2 | \qquad (\because \text{Re}\,(z) \leq |z|)$$

or $\leq |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$ $(\because |\bar{z}_2| = |z_2|)$ or $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

(i.e) $|z_1 + z_2| \leq (|z_1| + |z_2|)$ ---------- (1)

Q: The modulus of difference of two complex numbers is greater than or equal to the difference of their moduli.

Solution: Let z_1 and z_2 be the two complex numbers and their conjugates are $\bar{z}_1$ and $\bar{z}_2$. Then

$$|z_1 - z_2|^2 = (z_1 - z_2)\,\left(\overline{z_1 - z_2}\right) = (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\,\bar{z}_2)$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1\,\bar{z}_2| \qquad (\because \mathrm{Re}\,(z) \leq |z|)$$

and $-\mathrm{Re}(z) \geq -|z|$

$$\geq |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2| \qquad (\because |\bar{z}_2| = |z_2|)$$

or $|z_1 - z_2| \geq (|z_1| - |z_2|)$ ---------- (2)

Note: The inequalities (1) and (2) are important in future lessons on complex variables.

In coordinate geometry, the equation of a circle with origin as center and radius r is given by $x^2 + y^2 = r^2$. This can be represented in complex variables as $|z|^2 = r^2$ or simply $|z| = r$. Thus

$|z| = 1$ represents the equation of a unit circle with origin as centre. Generalizing this concept, $|z - \alpha| = r$ is the equation of circle with r units radius and centre at α (complex).

Some noteworthy points in understanding the circles are as follows.

$|z - \alpha| = r$: All the points on the circumference of the circle.

$|z - \alpha| < r$: All the points inside the circle.

$|z - \alpha| \leq r$: All the points within and on the circumference of the circle.

$|z - \alpha| > r$: All the points outside the circle.

5.3 Definitions:

5.3.1 Neighborhood of point:

It is the set of all points z such that $|z - z_0| < \in$ where $\in$ is an arbitrarily chosen small positive number. i.e., all points interior to $|z - z_0| = \in$ are called the neighborhood of z_0.

5.3.2 Limit:

Let f(z) be defined and single - valued. Let f(z) = u(x,y) + iv(x,y). We say that the number λ is limit of f(z) as z approaches z_0 and write $\underset{z \to z_0}{\text{Lt}} f(z) = \lambda$ if for any arbitrary small positive number ∈, we can find some positive number δ such that $|f(z) - \lambda| < \in$ for all values in $|z - z_0| < \delta$.

This means that the values of f(z) are as close as desired to λ for all z which are sufficiently close to z_0 as shown in figure 2.

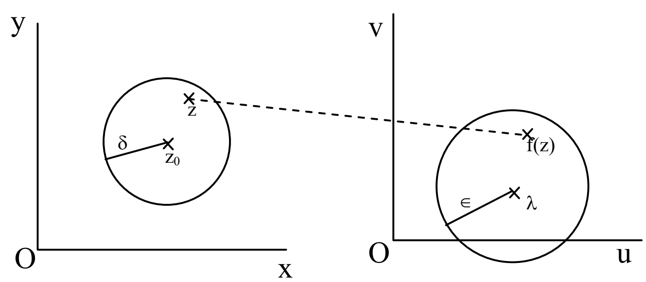


Fig 2: **Limit. Dotted line shows the correspondence between z approaching z_0 and f(z) approaching λ**

Note: The definition of a limit implies that in whatever manner z may approach z_0, the limit must be uniquely λ. Since z is a function of x and y (two dimension), z may approach z_0 along any

radius vector or any curve. Recalling our concept of a limit in one dimension, $\underset{x \to a}{\text{Lt}} f(x) = k$, it means that the limit from the left and the limit from the right should be equal for the uniqueness of the value k and there are no other paths.

5.3.3 Continuity:

A single valued function f(z) is continuous at the point z_0, if for a given arbitrarily small positive number ∈, there exists a number δ such that $| f(z) - f(z_0) | < \in$ for all z satisfying $|z - z_0| < \delta$ where δ depends on ∈.

This means that f(z) is continuous at z_0 if $\underset{Z \to Z_0}{\text{Lt}} f(z)$ uniquely exists in whatever manner z approaches z_0 and that value is the value of the function at z_0. Or $\underset{Z \to Z_0}{\text{Lt}} f(z) = f(z_0)$.

5.3.4 Derivatives:

If f(z) is single valued in some region of the z –plane, the derivative of f(z) is defined as

$$f'(z) = \underset{\Delta Z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{----------} \ (3)$$

provided that the limit exists in whatever manner Δz approaches zero. In such case we say that f(z) is differentiable at z.

5.3.5 Analytic functions:

A function f(z) which is single valued and differentiable at every point of a region, is said to be analytic in the region. The terms regular and holomorphic are sometimes used as synonyms for analytic.

1.3.6 Singular points:

A point at which f(z) fails to be analytic is called a singular point or singularity of f(z). We consider various types of singularities that exist at a latter stage.

Note: The practical approach in finding out the singular point is to find out the point where the given function becomes infinite.

5.4 Cauchy – Riemann equations:

Q: A necessary condition that w = f(z) = u (x, y) + iv (x, y) be analytic in a region R is that u and v satisfy the Cauchy – Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{----------} \ (4)$$

or $u_x = v_y$ and $u_y = - v_x$

In addition to the existence of the partial derivatives in (4), if they are also continuous, then the Cauchy – Riemann equations are sufficient conditions for f(z) to be analytic in R.

Solution:

Necessary:

If f(z) = u (x, y) + iv (x, y) is to be analytic, the limit

$$\underset{\Delta Z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

$$= \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{Lt} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\,\Delta y}$$

-------------- (5)

must exist in whatever manner Δz or (Δx and Δy) tends to zero. Let us consider two simple approaches

Case 1: In $\Delta z = \Delta x + i\,\Delta y$ approaching zero let us consider that $\Delta y = 0$ which means that $\Delta z = \Delta x$ (purely real). So Δz tending to zero means it approaches zero along the real axis. In such a case, (5) becomes

$$f'(z) = \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$= u_x + iv_x \quad \text{----------} \quad (6)$$

provided the partial derivative exist.

Case 2: If $\Delta x = 0$ and $\Delta y \to 0$, then $\Delta z = \Delta y$ (purely imaginary) tends to zero. So (5) becomes

$$f'(z) = \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\,\Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} \right]$$

$$= \frac{1}{i} u_y + v_y = -i\,u_y + v_y. \quad \text{----------} \quad (7)$$

Now f(z) cannot be analytic unless these two limits as in (6) and (7) must be identical. So the necessary condition that f(z) be analytic is

$u_x + i\,v_x = -\,i\,u_y + v_y$

Or $u_x = v_y$; $v_x = -\,i\,u_y$ ---------- (8)

Sufficient:

Apart from the existence of the partial derivatives in (8), since u_x and u_y are supposed continuous, we have

$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$

$= \{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)\} + \{u(x, y + \Delta y) - u(x, y)\}$

$= (u_x + \epsilon_1)\,\Delta x + (u_y + \eta_1)\,\Delta y$ by mean value theorem

$= u_x\,\Delta x + u_y\,\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$

where ϵ_1 and η_1 tend to zero as $\Delta x \to 0$ and $\Delta y \to 0$

Again, considering that v_x and v_y are supposed continuous, we get a similar expression for Δv as

$\Delta v = v_x \Delta x + v_y \Delta y + \epsilon_2 \Delta x + \eta_2\,\Delta y$ where ϵ_2, η_2 tend to zero as Δx and Δy tend to zero.

Then $\Delta w = \Delta u + i\,\Delta v$

$$= (u_x + i\,v_x)\,\Delta x + (u_y + i\,v_y)\,\Delta y + \epsilon \Delta x + \eta \Delta y \quad \text{----------} \quad (9)$$

Where $\epsilon = \epsilon_1 + i\,\epsilon_2 \to 0$ and $\eta = \eta_1 + i\,\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

If $\Delta w = f(z)$ satisfies Cauchy – Riemann equations then we have to prove that unique derivative of f(z) exists.

By Cauchy – Riemann equations, (9) takes the form

$\Delta w = (u_x + i\, v_x)\,\Delta x + (-v_x + i\, u_x)\,\Delta y + \in\Delta x + \eta\Delta y$

$= (u_x + i\, v_x)\,(\Delta x + i\,\Delta y) + \in\Delta x + \eta\Delta y$

Dividing with $\Delta z = \Delta x + i\,\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \underset{\Delta Z\to 0}{Lt}\frac{\Delta w}{\Delta z} = u_x + i\, v_x \text{ ---------- } (10)$$

so that the derivative exists and unique. That is f(z) is analytic.

5.5 Harmonic functions:

If a function having continuous second order partial derivatives satisfies Laplace's equation, then that function is called harmonic function.

Every analytic function f(z) satisfies Cauchy – Riemann equations

$$\frac{du}{dx} = \frac{dv}{dy} \; ; \; \frac{du}{dy} = -\frac{dv}{dx} \text{ ------------- } (8)$$

Differentiating the first equation of (8) partially w.r.t. x and the second equation w.r.t y and adding, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ ----------- } (11)$$

Similarly, differentiating first equation of (8) partially w.r.t. y and the second w.r.t. x and subtracting, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ ----------- } (12)$$

Thus in the analytic function

f(z) = u(x, y) + i v(x, y)

u and v satisfy Laplace equation and hence they are called harmonic functions. Further, two harmonic functions u and v are such that u + iv is an analytic function, then they are called **conjugate harmonic functions**. (This 'conjugate' term should not be confused with the complex conjugate of a complex number)

5.6 Examples:

(1) Show that $f(z) = \bar{z}$ is nowhere analytic

Solution: If $f(z) = \bar{z} = x - iy$, then

$$f(z + \Delta z) = \bar{z} + \overline{\Delta z} = (x - iy) + (\Delta x - i\Delta y)$$

$$\therefore \frac{d\bar{z}}{dz} = \underset{\Delta Z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{Lt} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Let Δx and Δy approach along the radius vector

$y = mx$. Then

$$\frac{d\bar{z}}{dz} = \underset{\Delta x \to 0}{Lt} \frac{\Delta x - i\,m\,\Delta x}{\Delta x + i\,m\,\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{1 - i\,m}{1 + i\,m}$$

$= \frac{1 - i\,m}{1 + i\,m}$. This value is not unique since m is an arbitrary constant. So $\frac{d\bar{z}}{dz}$ does not exist.

Hence it is nowhere analytic.

(2) Prove that the function $u + iv = f(z)$ where

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & (z \neq 0) \\ 0 & (z = 0) \end{cases}$$

Is continuous and that the Cauchy – Riemann equation are satisfied at the origin'

Yet $f'(0)$ = does not exist.

Solution: $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$

$$= \frac{x^3 - y^3}{x^2 + y^2} + i\,\frac{x^3 + y^3}{x^2 + y^2} \quad \text{from which}$$

$$u = \frac{x^3 - y^3}{x^2 + y^2} \text{ and } v = \frac{x^3 + y^3}{x^2 + y^2} \quad \text{when } z \neq 0.$$

Both u and v are rational and finite for all values of $z \neq 0$. Hence f(z) is continuous for all $z \neq 0$.

Now at $z = 0$, both u and v are zero. So they are continuous at the origin.

We know that

$$\frac{\partial u}{\partial x} = \underset{h \to 0}{Lt} \frac{u(x+h, y) - u(x, y)}{h}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{\substack{x=0 \\ y=0}} = \underset{h \to 0}{Lt} \frac{u(h, 0) - u(0, 0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{h - 0}{h} = 1 \qquad (\because \text{at } z = 0,\ f(0) = 0)$$

Similarly, it is seen that

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{h\to 0}{\text{Lt}} \frac{-h-0}{h} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{h\to 0}{\text{Lt}} \frac{h-0}{h} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{h\to 0}{\text{Lt}} \frac{h-0}{h} = 1$$

Thus at the origin $u_x = v_y$ and $u_y = -v_x$ (i.e) Cauchy – Riemann equation are satisfied. But the derivative at the origin is

$$f'(0) = \underset{Z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z-0}$$

$$= \underset{\substack{x\to 0\\y\to 0}}{\text{Lt}} \frac{x^3-y^3}{(x^2+y^2)(x+iy)} + i\,\frac{x^3+y^3}{(x^2+y^2)(x+iy)}$$

Since both numerator and denominator are homogeneous expressions of the same order, let x and y approach zero along any radius vector (i.e.) y = mx. Then

$$f'(0) = \underset{x\to 0}{\text{Lt}} \frac{1-m^3}{(1+m^2)(1+im)} + i\,\frac{1+m^3}{(1+m^2)(1+im)}$$

which is independent of x. Further, since m is arbitrary, $f'(0)$ is not unique and f(z) is continuous everywhere.

(3) Show that f(z) = $\sqrt{|xy|}$ is not analytic at the origin although Cauchy – Riemann equations are satisfied at that point.

Solution: Given that f(z) = $\sqrt{|xy|}$. Since |xy| is always a positive quantity, f(z) = $\sqrt{|xy|}$ is always real. So u(x, y) = $\sqrt{|xy|}$; v (x, y) = 0.

Now $$\left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{h\to 0}{\text{Lt}} \frac{u(h,0)-u(0,0)}{h} = \underset{h\to 0}{\text{Lt}} \frac{0-0}{h} = 0$$

Similarly $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are zeros. So Cauchy – Riemann equation are satisfied at the origin.

Now $f'(0) = \underset{Z\to 0}{Lt}\ \frac{f(z) - f(0)}{z - 0} = \underset{\substack{x\to 0\\ y\to 0}}{Lt}\ \frac{\sqrt{|xy|}}{x + iy}$

Since the numerator and denominator are homogeneous expressions of the same order, consider the radius vector y = mx along which x and y approach zero. Then

$f'(0) = \underset{x\to 0}{Lt}\ \frac{x\sqrt{|m|}}{x(1+im)} = \frac{\sqrt{|m|}}{(1+im)}$ which gives different values for different values of the arbitrary constant m. Hence $f'(0)$ is not unique or the derivative does not exist or f(z) is not analytic at the origin.

(4) Show that $w = x^2 - y^2 + 2\,i\,xy$ is everywhere analytic in the entire complex plane and express the derivative of w w.r.t z as a function of z alone.

Solution: Given that $w = (x^2 - y^2) + 2\,i\,xy$ in which $u = x^2 - y^2$ and $v = 2\,xy$

$\therefore u_x = 2x,\ u_y = -2y;\quad v_x = 2y,\ v_y = 2x$

(i.e.) Cauchy – Riemann equations are identically satisfied in the complex plane. More over the first order partial derivatives are everywhere continuous. So the derivative $\frac{dw}{dz}$ should exist according to the sufficient condition for the analytic functions and it is given by

$$\frac{dw}{dz} = u_x + iv_x \quad \text{----------} \quad (10)$$

$$= 2x + 2iy = 2(x + iy) = 2z.$$

(5) In any analytic function w = u(x, y) + iv(x, y), if x and y are replaced by their equivalents,

$x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

then w will appear as a function of z alone

Solution: Although z and $\bar{z}$ are clearly dependent, w can be formally considered as a function two new independent variables z and $\bar{z}$. Then, if w has to appear as a function of z only, we have to prove that $\frac{\partial w}{\partial \bar{z}}$ is identically zero.

Now

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial(u+iv)}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\,\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

But from the expression of x and y in terms of z and $\bar{z}$,

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \quad ; \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2}$$

So

$$\frac{\partial w}{\partial \bar{z}} = \left(\frac{1}{2}u_x + \frac{i}{2}u_y\right) + i\left(\frac{1}{2}v_x + \frac{i}{2}v_y\right)$$

$$= \frac{1}{2}\left(u_x - v_y\right) + \frac{i}{2}\left(u_y + v_x\right) = 0$$

($\because$ w = u + iv is given as analytic and so u and v satisfy Cauchy –Riemann equation)

(i.e.) w is a function of z alone.

(6) Find v of the analytic function f(z) = u + iv if $u = e^{-x}(x \sin y - y \cos y)$. Express f(z) as a function of z.

Solution: Since f(z) is analytic, it should satisfy Cauchy – Riemann equations. So

$v_y = u_x = e^{-x} \sin y - x\, e^{-x} \sin y + y\, e^{-x} \cos y$ ---------- (13)

$v_x = -u_y = e^{-x} \cos y - x\, e^{-x} \cos y - y\, e^{-x} \sin y$ ---------- (14)

Integrating (13) partially w.r.t y, we get

$v = -\, e^{-x} \cos y + x\, e^{-x} \cos y + e^{-x}(y \sin y + \cos y) + G(x)$

$= y\, e^{-x} \sin y + x\, e^{-x} \cos y + G(x)$ ---------- (15)

where G(x) is an arbitrary real function of x.

substituting (15) in (14), we get

$-y\, e^{-x} \sin y - x\, e^{-x} \cos y + e^{-x} \cos y + G'(x)$

$= -y\, e^{-x} \sin y - x\, e^{-x} \cos y - y\, e^{-x} \sin y$

or $G'(x) = 0$ i.e., $G(x) = k$ (a constant)

So (15) gives $v = y\, e^{-x} \sin y + x\, e^{-x} \cos y + k$ ---------- (16)

$\therefore f(z) = u + iv$

$$= e^{-x}\left\{x\frac{e^{iy}-e^{-iy}}{2i} - y\frac{e^{iy}+e^{-iy}}{2}\right\} + i\,e^{-x}\left\{y\frac{e^{iy}-e^{-iy}}{2i} + x\frac{e^{iy}+e^{-iy}}{2}\right\} + ik$$

$$= i\,(x+iy)\,e^{-(x+iy)} + ik = iz\,e^{-z} + ik$$

Some more methods of finding f(z) as a function of z**:**

Method 1:

We have $f(z) = f(x+iy) = u(x, y) + iv(x, y)$

Putting $y = 0$, $f(x) = u(x, 0) + iv(x, 0)$

Replacing x by z, $f(z) = u(z, 0) + iv(z, 0)$

In the given problem $u(z, 0) = 0$, $v(z, 0) = z\,e^{-z}$

$\therefore f(z) = iz\,e^{-z}$ apart from an additive constant.

Method 2:

Let us put $\frac{\partial u}{\partial x} = u_1(x, y)$ and $\frac{\partial u}{\partial y} = u_2(x, y)$. From the sufficient condition of the analytic function f(z), it is seen that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ ---------- (10)

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

or $f'(x+iy) = u_1(x, y) - i\,u_2(x, y)$

Putting $y = 0$, $f'(x) = u_1(x, 0) - i\,u_2(x, 0)$

Replacing x by z, $f'(z) = u_1(z, 0) - i\,u_2(z, 0)$

Or $f'(z) = 0 - i\,(z\,e^{-z} - e^{-z}) = -i\,(z\,e^{-z} - e^{-z})$ from (13) and (14)

Integrating w.r.t. z, $f(z) = i\,z\,e^{-z}$ apart from the integrating constant.

By separating this into real and imaginary parts, we get $v = e^{-x}(y \sin y + x \cos y)$ apart from a constant.

Note: This is a method for obtaining f(z) without finding the conjugate function.

Method 3:

After finding v(x, y) and substituting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ in $f(z) = u(x, y) + iv(x, y)$ one can find f(z) as a function of z alone after a tedious procedure.

(7) If u and v are conjugate harmonic functions, show that v and – u as well as –v and u are also conjugate harmonic functions, but that v and u are not.

Solution: Here we have to consider these conjugate functions as real and imaginary parts in their order for the complex function.

Given that f(z) = u + iv is an analytic function

$$\therefore \left.\begin{aligned} u_z &= v_y \\ u_y &= v_x \end{aligned}\right\} \text{Cauchy – Riemann equation.}$$

Then v – iu is analytic if $v_x = - u_y$ and $v_y = u_z$

Similarly –v + iu is analytic if $-v_x = u_y$ and $-v_y = -u_x$

Which are true from the above Cauchy – Riemann equation.

However v + iu is not analytic as $v_x = u_y$ and $v_y = -u_x$ are not the same as the Cauchy – Riemann equations.

(8) If ω = u(x, y) + iv(x, y) is an analytic function of z, then the curves of the family u(x, y) = c are the orthogonal trajectories of the curves of the family v(x, y) = k and vice versa.

Solution: Since u(x, y) = c

$$\therefore \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0$$

or
$$\frac{dy}{dx} = -\frac{u_x}{u_y} = +\frac{v_y}{v_x} \qquad \text{(due to Cauchy – Riemann equation.)}$$

Again with the trajectories v(x, y) = k, we get

$$\frac{dy}{dx} = -\frac{v_x}{v_y}$$

If the two trajectories are orthogonal, the product of the two slopes must be equal to –1. Thus, from the slopes obtained, it is seen that $\left(\frac{v_y}{v_x}\right) \text{x} \left(-\frac{v_x}{v_y}\right) = -1$

Hence the result.

5.7 Summary:

This lesson, starting with an introduction, projects the rudiments of complex numbers and functions. Then the basic definitions of certain parameters already familiar in real analysis are given with respect to complex region. Uniqueness of the limit is highlighted which can be appreciated while dealing with the derivation of Cauchy – Riemann conditions. The equation of circle and inequalities in the complex plane, play important role in future theorems and problems.

The definition of an analytic function is given and the necessary and sufficient conditions for a function to be analytic are derived. The real and imaginary parts of every analytic function are seen to be harmonic functions (conjugates) satisfying Laplace equation.

Typical and assorted problems have been worked and questions given at the end of the lesson.

5.8 Key Terminology:

Argand diagram — mod.z — amp.z — polar form — single valued function — neighborhood — limit — continuity — differentiability — Analytic functions — Cauchy – Riemann equations — Harmonic functions.

5.9 Self – assessment questions:

1. If f(z) = u + iv is an analytic function where u^2+v^2 is a constant, show that f(z) is a constant.
2. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is a harmonic function. Find its harmonic conjugate of the analytic function f(z) = u+iv and determine f(z) as a function of z.
3. Show that $w = z\bar{z}$ is everywhere continuous and it is nowhere analytic except at the origin.
4. If $z = re^{i\theta}$, show that the Cauchy – Riemann equations take the form

   $$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$
5. If f(z) and $\overline{f(z)}$ are both analytic functions show that f(z) is a constant.
6. If f(z) = u + iv is an analytic function where v = amp.z, show that v is a harmonic function and find f(z) .
7. In the analytic function f(z) = u + iv, $u = \ln(x^2 + y^2)$ obtain f(z) as a function of z without finding the harmonic conjugate of u.
8. Determine the analytic function f(z) = u + iv when $u + v = x^2 - y^2 + 2xy$.

5.10 Reference Books:

1. M.R. Spiegel ‘Complex variables’, McGraw – Hill Book co., 1964.
2. E. Kreyszig ‘Advanced engineering mathematics’, Wiley Eastern Pvt., Ltd., 1971.
3. B.D. Gupta ‘Mathematical Physics’, Vikas publishing House, Sahibabad, 1980.

Unit - II

Lesson - 6

COMPLEX INTEGRATION

Objective of the lesson :

* To give the definition of integration in complex variables
* To evaluate certain basic integrals from the definition of integral
* To explain the important concepts of simply and multiply connected region.
* To prove Cauchy's theorem on analytic function.
* To derive Cauchy's integral formula both on simply and multiply connected region.
* To prove derivatives theorem on analytic function
* To prove the converse of Cauchy's theorem

Structure of the lesson :

6.1 Introduction

The definition of an integral in complex variables runs on similar lines as in real analysis. The importance is stressed in the evaluation of integrals around closed contours. Cauchy's theorem is given to make certain integrals easy for evaluation. The values of the complex functions and their derivatives at given points are expressed interms of the integrals containing those functions. Examples for further understanding are given.

6.2 Some Definitions :

If a point on arc is such that $z = \phi(t) + i\psi(t)$ and if ϕ and ψ are real continuous functions of the real variable 't' defined in the range $\alpha \le t \le \beta$, then the arc is called a continuous arc.

If z is satisfied by more than one value of t in the given range, then the point z is a multiple point of the arc.

A continuous curve without multiple points or which does not interesect itself is called a Jordan Curve.

A continuous Jordan curve made up of a finite number of regular arcs is called a contour.

6.3 Complex line integrals (Riemann's definition of integration) :

Let a function $f(z)$ of complex variable z be a continuous function defined along the curve C with end points A and B as shown in Fig. 1 Let $Z_0 = a, Z_1, Z_2, ...Z_n = t$ be a mode of subdivision of the curve C.

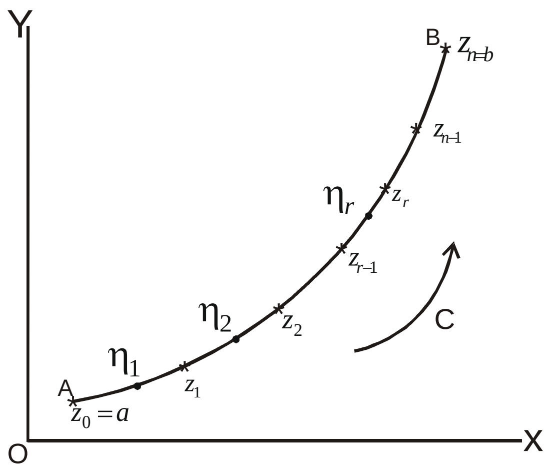


Fig. 1

Let $\eta_1, \eta_2, ..., \eta_n$ be the set of points in the n subdivisions of the curve C such that η_1 lies on the arc $Z_0 Z_1, \eta_2$ lies on $Z_1 Z_2$ and so on.

Now form the sum $\sum_{r=1}^{n}\left(Z_r - Z_{r-1}\right) f\left(\eta_r\right)$ where $Z_{r-1} \le \eta_r \le Z_r$.

When the number of subdivisions is made inifinite, then the limit of the sum, if exists uniquely for any path whatsoever joining a to b, is called the integral of $f\left(z\right)$ over C from a to b. It is written as

$$\underset{n\to\infty}{Lt} \sum_{r=1}^{n}\left(Z_r - Z_{r-1}\right) f\left(\eta_r\right) = \int_a^b f\left(z\right)dz .$$

Note : *When the integral is taken around a closed contour, the traversal along the closed path in the counter clockwise direction is conventionally taken as positive direction.*

6.4 Concepts on Basic Integrals :

Q. Evaluate $\int_C z\,dz$ with the help of the definition where C is

(i) from a to b (ii) Closed

Solution : (i) By definition,

$$\int_C f\left(z\right)dz = \underset{n\to\infty}{Lt} \sum_{r=1}^{n}\left(Z_r - Z_{r-1}\right) f\left(\eta_r\right)$$

Here $\int_C z\ dz = \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \eta_r \left(Z_r - Z_{r-1}\right)$

η_r is any point in the very small interval Z_{r-1} to Z_r when $n \to \infty$. In such a case, η_r can be taken as equal to Z_{r-1} or Z_r or any other point in between Z_{r-1} and Z_r .

So when $\eta_r = Z_{r-1}$, the above integral reduces to

$$\int_C f\left(z\right)dz = \underset{n\to\infty}{Lt} \sum_{r=1}^{n} Z_{r-1}\left(Z_r - Z_{r-1}\right) \text{ .... (1)}$$

Similarly, when $\eta_r = Z_r$, then

$$\int_C f\left(z\right)dz = \underset{n\to\infty}{Lt} \sum_{r=1}^{n} Z_r\left(Z_r - Z_{r-1}\right) \text{ .... (2)}$$

Now $\frac{1}{2}\left[(1)+(2)\right]$ gives

$$\int_c f(z)dz = \frac{1}{2}\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left[Z_{r-1}\left(Z_r - Z_{r-1}\right) + Z_r\left(Z_r - Z_{r-1}\right)\right]$$

$$= \frac{1}{2}\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left(Z_r^2 - Z_{r-1}^2\right)$$

$$= \frac{1}{2}.\underset{n\to\infty}{Lt}\left(Z_n^2 - Z_0^2\right)$$

$$= \frac{1}{2}.\left(b^2 - a^2\right) \text{ because } Z_0 = a \text{ and } Z_n = b$$

(ii) If C is a closed contour, the starting point Z_0 = a coincides with the end point Z_n = b in which case

$$\oint_c z\ dz = 0$$

Q. Starting from definition evaluate $\int dz$ where C is (i) open (ii) closed.

Solu: **(i)** From the definition, we have

$$\int_C f(z)dz = \underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left[f(\eta_r)\left(Z_r - Z_{r-1}\right)\right]$$

Here $f(z) = 1$, So $\int_C dz = \underset{n\to\infty}{Lt}\sum_{r=1}^{n} 1.\left(Z_r - Z_{r-1}\right)$

or
$$\int_C dz = \underset{n\to\infty}{Lt}\left[\left(Z_1 - Z_0\right) + \left(Z_2 - Z_1\right) + \ldots\ldots\ldots + \left(Z_n - Z_{n-1}\right)\right]$$

$$= \underset{n\to\infty}{Lt}\left(Z_n - Z_0\right)$$

$$= (b-a) = \text{ Chord ab } \because Z_0 = a,\ Z_n = b$$

(ii). If the curve is closed so that Z_0 and Z_n coincide, then $\oint_C dz = 0$

Q. If $f(z)$ is integrable along a curve C having a finite length L and if there exists a positive number M such that $|f(z)| \le M$ on C, then $\left|\int_C f(z)dz\right| \le ML$

Solution : By definition,

$$\int_C f(z)dz = \underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left[f(\eta_r)\left(Z_r - Z_{r-1}\right)\right]$$

$$\left|\int_C f(z)\,dz\right| = \underset{n\to\infty}{Lt}\left|\sum_{r=1}^{n}\left[f(\eta_r)\left(Z_r - Z_{r-1}\right)\right]\right|$$

$$\leq \underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left|f(\eta_r)\right|\left|\left(Z_r - Z_{r-1}\right)\right|$$

$$\leq M\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left|Z_r - Z_{r-1}\right|$$

$\leq ML$ (since $\left|Z_r - Z_{r-1}\right|$ is the chord length same as arc length as n is tending to ∞, the sum of these arc lengths is the length of the curve (i.e.) L)

6.5 Simply and Multiply Connected Regions :

(i) Simply connected region :

A region R is called simply connected if every closed curve C in R can be continuously shrunk to any point in R without leaving R as shown in Fig. 2.

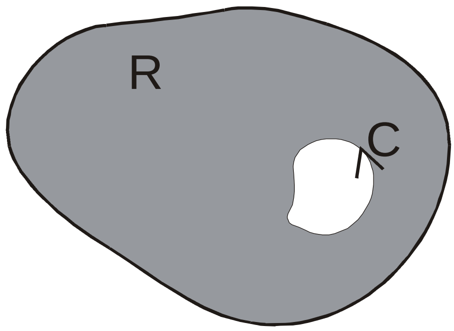


Fig. 2.

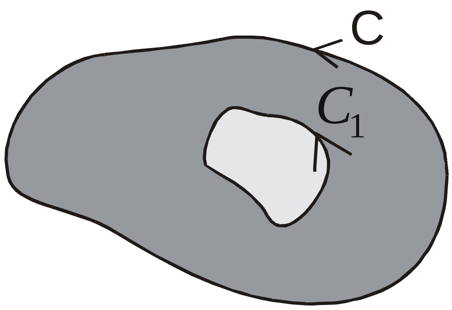
OR

The interior of a closed curve which has no self-intersections is the simply connected region.

The simply connected region has only one boundary i.e., The external boundary.

(ii) Multiply connected Regions :

A region R which is not simply connected is called multiply connected region. A doubly connected region has one external boundary and one internal boundary. A triply connected region has two internal boundaries apart from one external boundary as shown in Fig. 3.



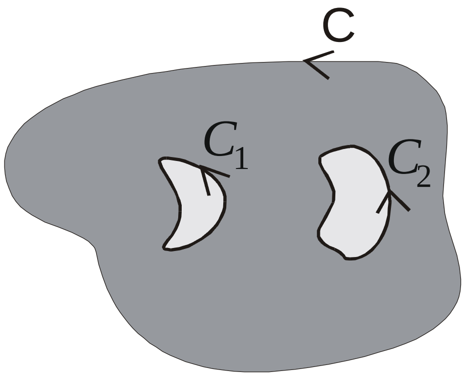


Fig. 3. (a) Doubly connected region (C-external, and C_1- internal boundaries)

(b) Triply connected region (C-external and C_1 and C_2 being internal boundaries)

(iii) Change of multiply connected regions into simply connected regions :

A multiply connected region is converted into a simply connected region as shown in Fig. 4 by making cross cuts.

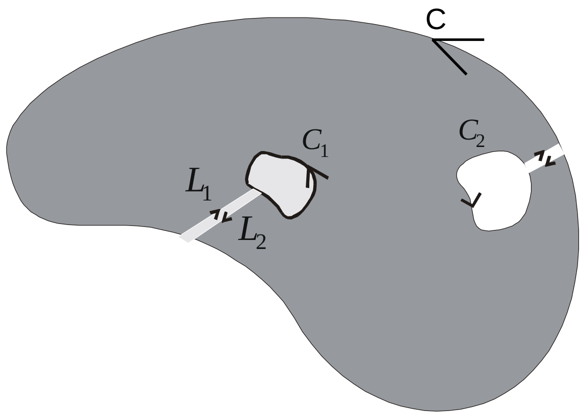


Fig. 4

A thin cross cut should be made from the outer boundary to the inner boundary in the shortest length possible. As there are three boundaries in the multiply connected region (Fig. 3b), the connected region with cross cuts in Fig. 4 has only one boundary and hence it is a simply connected region.

Thus the theorems, which are true for simply connected regions are also true for multiply connected regions as they can be converted into simply connected regions by making cross cuts.

6.6 Cauchy's theorem (Cauchy's integral theorem) :

Theorem : If $f(z)$ is an analytic function of z and if $f'(z)$ is continuous at each point within and on a closed contour C, then $\int_C f(z)dz = 0$.

Proof : We know the Green's theorem in a plane that if $P(x,y), Q(x,y), \frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are all continuous function of x and y in the domain 'D', then

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad ....\ (1)$$

Let the given function be $f(z) = u(x,y) + i\text{v}(x,y)$ where $z = x + iy$ and $dz = dx + i\,dy$

$$\therefore \int_C f(z)dz = \int_C (u + i\text{v})(dx + i\,dy)$$

$$= \int_C (u\,dx - \text{v}\,dy) + i\int_C (\text{v}\,dx + u\,dy)$$

$$= \iint_D (-\text{v}_x - u_y)\,dx\,dy + i\iint_D (u_x - \text{v}_y)\,dx\,dy \quad \text{by (1)}$$

= 0 by Cauchy - Riemann equations since $f(z)$ is analytic.

Note : ***(i)*** *Without asuiming the continuity of $f'(z)$, the theorem can be proved. (i.e.) If $f(z)$ is analytic evrywhere within and on the boundary of the closed contour C, then $\int_C f(z)\,dz = 0$. This theorem is named as Cauchy - Goursat Theorem whose proof is not necessary here. However, in future, whenever $f(z)$ is analytic everywhere in the region, we apply its implication that $\int_C f(z)\,dz = 0$.*

(ii) *If $f(z)$ is analytic in a region bounded by two simple closed curves C and C_1 (sense of direction being positive) i.e., in a doubly connected region (Fig. 3a), even then the Cauchy's integral theorem holds good. The argument runs as follows. The doubly converted region (Fig. 3a) can be connected into a simply connected region as shown in Fig. 4 by making cross cut. In such a region, the boundary being $CL_1\ C_1L_2$, Cauchy's integral Theorem is applicable.*

$$\text{So}\left[\oint_C + \int_{L_1} - \oint_{C_1} + \int_{L_2}\right] f(z)\,dz = 0$$

or $\oint_C f(z)dz = \int_{C_1} f(z)dz$ .... (2)

Since L_1 and L_2 are the same integrals with opposite sense of direction and hence $\int_{L_1} + \int_{L_2}$ is zero.

The concept of (2) is important in future. In n-ply connected region, similar equation to (2) can be written as

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + ... \oint_{C_{n-1}} f(z)dz$$

6.7 Cauchy's Integral formula :

Statement : If $f(z)$ is analytic inside and on the boundary of 'C' of a simply connected region and if 'a' is any point interior to 'C', then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \text{ .... (3)}$$

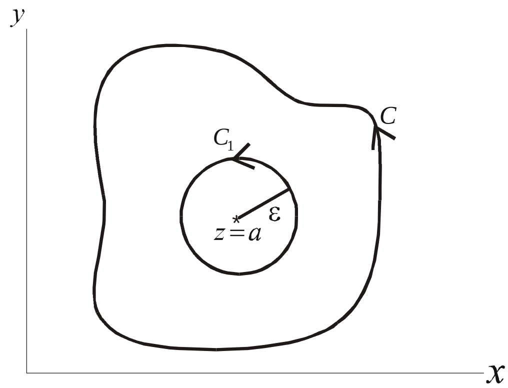


Fig. 5

Proof : The function $\frac{f(z)}{z-a}$ is everywhere analytic in C except at the singularity Z = a. If we remove the singularity by encircling with $C_1 : |Z-a| = \varepsilon$ as shown in Fig. 5, then in the doubly connected region between the closed curves 'C' and 'C_1', the function $\frac{f(z)}{z-a}$ is everywhere analytic. So, according to Cauchy's integral theorem,

$$\int_C \frac{f(z)}{z-a}\,dz - \int_{C_1} \frac{f(z)}{z-a}\,dz = 0$$

or
$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz + \int_{C_1} \frac{f(a)}{z-a}\,dz$$

$$= I_1 + I_2 \text{ .... (4)}$$

$$I_1 : |I_1| = \left| \int_{C_1} \frac{f(z)-f(a)}{z-a} dz \right| \le \int_{C_1} \frac{|f(z)-f(a)|}{|z-a|} |dz|$$

$$\le \int_0^{2\pi} \frac{\eta\,\varepsilon\, d\theta}{\varepsilon} \quad \left[\because z-a = \varepsilon e^{i\theta},\ dz = \varepsilon\, e^{i\theta}\, i\, d\theta \right.$$

$|dz| = \varepsilon\, d\theta$ and $f(z)$ is a continuous at z = a, and so

(i.e.) $\le 2\pi\eta$ $\quad \left. |f(z) - f(a)| < \eta \text{ for all } |z-a| < \varepsilon \right]$

$$\therefore\ I_1 \to 0$$

$$I_2 : \int_{C_1} \frac{f(a)}{z-a}\,dz = f(a) \int_0^{2\pi} \frac{\varepsilon\, e^{i\theta}\, i\, d\theta}{\varepsilon\, e^{i\theta}}$$

$$= 2\pi i\ f(a)$$

$\therefore$ (4) becomes $\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$.

Hence the result.

6.8 Cauchy's integral formula for multiply connected regions :

We consider a doubly connected region between the two closed curves C and C_1 as in Fig. 3a in which $f(z)$ is given to be analytic. Let a be the interior point of this doubly connected region. Then the function $\frac{f(z)}{z-a}$ has a singularity in the doubly connected region. If we eliminate this singularity by encircling with $\Gamma : |z-a| = \varepsilon$, then in the resulting triply connected region whose boundaries are C, C_1 and Γ, $\frac{f(z)}{z-a}$ is everywhere analytic. By changing that region to simply connected region and using Cauchy's integral theorem we can write

$$\int_C \frac{f(z)}{z-a}\,dz - \int_{C_1} \frac{f(z)}{z-a}\,dz - \int_\Gamma \frac{f(z)}{z-a}\,dz = 0$$

$\int_\Gamma \frac{f(z)}{z-a}\,dz$ is already proved to be lie $2\pi i\ f(a)$.

$$\therefore f(a) = \frac{1}{2\pi i}\left[\int_C \frac{f(z)}{z-a}\,dz - \int_{C_1} \frac{f(z)}{(z-a)}\,dz\right]$$

If this is extended to n-ply connected region then

$$f(a) = \frac{1}{2\pi i}\left[\int_C \frac{f(z)}{z-a}\,dz - \int_{C_1} \frac{f(z)}{z-a}\,dz \ldots\ldots\ldots\ldots \int_{C_{n-1}} \frac{f(z)}{z-a}\,dz\right]$$

6.9 Derivatives theorem on analytic functions :

Theorem : If $f(z)$ is analytic and has, at any interior point a, derivatives of all orders, then

Proof : Let us first prove for first and second order derivatives and extend it to the nth order, Now, we know that

$$f^{(n)}(a) = \frac{\underline{|n}}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \quad \ldots\ldots\ldots\ldots (5)$$

$$f'(a) = \underset{h\to 0}{Lt}\ \frac{f(a+h) - f(a)}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{1}{2\pi i}\frac{1}{h}\int_C \left(\frac{f(z)}{z-a-h} - \frac{f(z)}{z-a}\right)dz$$ [applying Cauchy's integral formula

since 'a' and 'a+h' are interior points.]

$$= \frac{1}{2\pi i}\ \underset{h\to 0}{Lt}\ \int_C \frac{1}{h}\ f(z)\frac{h}{(z-a)(z-a-h)}dz$$

$$= \frac{1}{2\pi i}\ \underset{h\to 0}{Lt}\int_C \frac{f(z)}{(z-a)^2}\cdot\frac{z-a}{z-a-h}\,dz$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{f(z)}{(z-a)^2}.\left[1+\frac{h}{z-a-h}\right]dz$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{f(z)}{(z-a)^2}\,dz+\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{h\,f(z)}{(z-a)^2(z-a-h)}dz \quad \text{......... (6)}$$

Let $I=\int_C \frac{h\,f(z)\,dz}{(z-a)^2\,(z-a-h)}$

Since $f(z)$ is analytic, it is bounded and $|f(z)|\le M$,

$$z-a=\varepsilon\,e^{i\theta},\ |z-a|=\varepsilon,\quad dz=\varepsilon e^{i\theta}.\,i\,d\theta,\ \ |dz|=\varepsilon\,d\theta$$

$$|z-a-h|\ge|z-a|-|h| \text{ or } \frac{1}{|z-a-h|}\le\frac{1}{\varepsilon-|h|}$$

$$\therefore |I|\le\int_0^{2\pi}\frac{|h|\,M\,\varepsilon\,d\theta}{\varepsilon^2(\varepsilon-|h|)}\le\frac{|h|\,M.\,2\pi}{\varepsilon(\varepsilon-|h|)}$$

So $I\to 0$ as $h\to 0$

$\therefore$ (6) becomes $f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$ .... (7)

Similarly the second order derivative of $f(z)$ at z = a may be obtained as

$$f''(a)=\underset{h\to 0}{Lt}\frac{f'(a+h)-f'(a)}{h}$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{1}{h}\left[\frac{1}{(z-a-h)^2}-\frac{1}{(z-a)^2}\right]f(z)\,dz \qquad \text{applying equation (7)}$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{1}{h}\,\frac{h[2(z-a)-h]}{(z-a)^2\,(z-a-h)^2}\,f(z)dz$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{(z-a)\left[2(z-a-h)+h\right]}{(z-a)^3(z-a-h)^2} f(z)dz$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{1}{(z-a)^3}\frac{\left[(z-a-h)+h\right]\left[2(z-a-h)+h\right]}{(z-a-h)^2} f(z)\,dz$$

$$=\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C \frac{1}{(z-a)^3}\frac{2(z-a-h)^2+3h(z-a-h)+h^2}{(z-a-h)^2} f(z)\,dz$$

$$=\frac{1}{2\pi i}\int_C \frac{2}{(z-a)^3} f(z)\,dz+\frac{1}{2\pi i}\underset{h\to 0}{Lt}\int_C f(z)\,h\,R(z)\,dz$$

Where $R(z)$ is bounded on 'C', so that $\int_C f(z)\,h\,R(z)\,dz$ tends to zero as $h\to 0$.

Thus $f''(a)=\frac{\lfloor 2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$ .... (8)

Similarly, assuming (5) to be valid for n = m, we can prove in a similar manner that it holds good for n = m+1. Hence we have, in general,

$$f^{(n)}(a)=\frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \quad \text{.......... (5)}$$

6.10 Morera's Theorem (Converse of Cauchy's Integral Theorem)

Statement : If $\int f(z)\,dz$, where $f(z)$ is continuous in the region, is zero when taken round any simply closed curve, then $f(z)$ is analytic.

Proof : Take z_0 as a fixed point and z any movable point in the given region as shown in the Fig. 6. Then the value of the integral

$$\int_{z_0}^{z} f(t)dt = F(z) \text{ (say) .... (6)}$$

is independent of the curve joining z_0 to z and is dependendt on z only.

So $F(z+h)=\int_{z_0}^{z+h} f(t)dt$ ....(7)

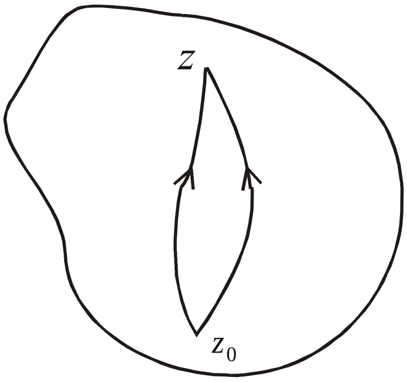


Fig. 6

Then $F(z+h)-F(z) = \int_{z_0}^{z_0+h} f(t)dt - \int_{z_0}^{z} f(t)dt$

$$= \int_{z}^{z+h} f(t)\, dt$$

Since the path of integration is independent of the curve joining z to z+h, let the path be a straight line so that

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(t)\, dt - \frac{1}{h} f(z)\int_{z}^{z+h} 1\, dt$$

$$= \frac{1}{h}\int_{z}^{z+h} \left[f(t) - f(z)\right] dt$$

$$\therefore \left| \frac{F(z+h)-F(z)}{h} - f(z) \right| = \left| \frac{1}{h}\int_{z}^{z+h} \left[f(t) - f(z)\right] dt \right|$$

$$\le \frac{1}{|h|}\int_{z}^{z+h} \left| f(t) - f(z)\right| |dt|$$

$$\le \frac{1}{|h|}\,\varepsilon . |h| = \varepsilon$$

So $\quad \underset{h\to 0}{Lt}\, \frac{F(z+h)-F(z)}{h} = F'(z) = f(z)$

showing that $F'(z)$ exists for all 'z' or $F'(z)$ is analytic in the region. That is, $f(z)$ analytic.

6.11 Examples

(1). If 'C' is a circle of radius 'r' and centre z_0 and if n is an integer, what is the value of

$$\int_C \frac{dz}{(z-z_0)^{n+1}}.$$

Solution : The given equation of the circle 'C' is $|z-z_0| = r$ or $(z-z_0) = r\, e^{i\theta}$

$\therefore dz = r\, e^{i\theta}\ i\, d\theta$ where θ ranges form 0 to 2π.

$$\therefore \int_C \frac{dz}{(z-z_0)^{n+1}} = \int_0^{2\pi} \frac{r\, e^{i\theta}\ i\, d\theta}{r^{n+1}\, e^{i(n+1)\theta}}$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta}.\, d\theta = \frac{i}{r^n} \left. \frac{e^{-in\theta}}{-in} \right|_0^{2\pi}$$

$$= \frac{-1}{nr^n}\left(e^{-2\pi ni} - 1\right) = \frac{1-e^{-2\pi ni}}{nr^n} = 0 \qquad \text{if } n \neq 0$$

If n = 0, we have to apply L' Hospital's rule to $\frac{1-e^{-2\pi ni}}{nr^n}$. Instead, it will be easy to start the problem afresh with n = 0.

$$\therefore \int_C \frac{dz}{(z-z_0)^1} = \int_0^{2\pi} \frac{r\, e^{i\theta}.\, i\, d\theta}{r\, e^{i\theta}} = 2\pi i$$

(2). Find the values of $\int_C \frac{e^z}{z^2+1} dz$ if C is a circle of unit radius with centre at (a) z = i and (b) z = -i.

Solution : (a) The integrand of the given integral has the singularities at $z = \pm i$ obtained by putting $z^2+1=0$. But with respect to the given circle $|z-i| = 1$, the singularity $z = +i$ only lies inside the contour. So we write the given integral to have a comparison with Cauchy's integral formula, in the form $\int_C \frac{e^z / (z+i)}{z-i} dz$ wherein $\frac{e^z}{z+i} = f(z)$ is everywhere analytic in the given circle and the entire integrand has a singularity at z = i w.r.t. the given circle. So according to Cauchy's integral formula

$$\int_C \frac{e^z/z+i}{z-i}\,dz = \left[\frac{e^z}{z+i}\right]_{z=i} 2\pi i = \frac{e^i}{2i} 2\pi i = \pi e^i$$

(b) In the given contour, $|z+i| = 1$, $z = -i$ singularity alone will lie inside. In such a case we write the integral as

$$\int_C \frac{e^z/z-i}{z+i}\,dz = \left[\frac{e^z}{z-i}\right]_{z=-i} \times 2\pi i = \frac{e^{-i} \times 2\pi i}{-2i} = -\pi e^{-i}$$

according to Cauchy's integral formula.

(3). What is the value of $\int_C \frac{z+1}{z^3-2z^2}dz$ where C is

(a) $|z-1-2i| = 2$ **(b)** $|z-2-i| = 2$ and **(c)** $|z| = 1$

Solution : (a) The integrand of the given integral has the singularities given by the roots of $z^3 - 2z^2 = 0$ as z = 0 (2nd order) and z = 2(first order).

But the distance of z = 0 (0, 0) from the centre 1 + 2i (1, 2) is greater than the radius 2 and hence z = 0 singularity lies outside the contour.

Similarly it can be seen that z = 2 lies outside the contour. Though the integrand has singularities, as far as the given contour is concerned, it is everywhere analytic. So, by Cauchy's integral theorem the given integral vanishes.

(b) The singularity z = 2 only lies in the contour $|z-2-i| = 2$, so the integral, to compare with the integral formula, can be written as $\int_C \frac{(z+1)/z^2}{(z-2)} dz$. Hence its value is given as

$$2\pi i \left[\frac{z+1}{z^2}\right]_{z=2} = 2\pi i \frac{3}{4} = \frac{3\pi i}{2}$$

(c) Out of the singularities z = 0 and z = 2 of the integrand, z = 0 only lies inside the given contour $|z| = 1$. But z = 0 is a second order singularity (since z^2 occured in the denominator, z = 0 is a second order singularity). Then the integral can be written as $\int_C \frac{(z+1)/(z-2)}{z^2} dz$ which comes under the applications of the derivatives theorem on analytic functions as

$$\int_C \frac{f(z)}{(z-a)^2}\, dz = 2\pi i\, f'(a) \quad \text{.......... (7)}$$

Here $f(z) = \dfrac{z+1}{z-2}$; a = 0

$$\therefore \int_C \frac{(z+1)/(z-2)}{z^2}\, dz = 2\pi i \left[\frac{d\left(\frac{z+1}{z-2}\right)}{dz} \right]_{z=0}$$

$$= 2\pi i \left[\frac{-3}{(z-2)^2} \right]_{z=0} = -\frac{3\pi i}{2}$$

(4). Show that $\int_C \frac{e^{xz}}{z^{n+1}}\, dz = 2\pi i \frac{x^n}{\lfloor n}$

Where C is any simple closed curve encircling the origin.

Solution : The integrand has a multiple order singularity at z = 0 whose order is n + 1 ($\because z^{n+1} = 0$ its order is n + 1) and it lies inside the contour.

So applying the derivatives theorem (5) on analytic functions

$$\int_C \frac{e^{xz}}{z^{n+1}}\, dz = \frac{2\pi i}{\lfloor n} \left[\frac{d^n e^{xz}}{dz^n} \right]_{z=0}$$

$$= \frac{2\pi i}{\lfloor n} . x^n$$

(5). If $f(z)$ is analytic within and on a circle of radius r with centre at z_0, then show that

$$\left| f^{(n)}(z_0) \right| \le \frac{\lfloor n M}{r^n}$$

Where 'M' is the maximum value of $|f(z)|$ on 'C'.

Solution : According to the derivatives theorem on analytic functions (5), we have

$$\left| f^{(n)}(z_0) \right| = \left| \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)^{n+1}} \right|$$

$$\leq \frac{\underline{|n}}{2\pi} \int_C \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{\underline{|n}}{2\pi} \frac{M}{r^{n+1}} \int_C |dz| \qquad \text{Since } z-z_0 = re^{i\theta},\ |z-z_0| = r,\ dz = re^{i\theta}.\,i\,d\theta$$

$$\leq \frac{\underline{|n}}{2\pi} \frac{M}{r^{n+1}} \int_0^{2\pi} r\,d\theta \qquad |dz| = r\,d\theta$$

i.e., $$\leq \frac{\underline{|n}}{2\pi} \frac{M}{r^{n+1}}\, 2\pi r$$

or $$\leq \frac{\underline{|n}\, M}{r^n}$$

This inequality is also called as Cauchy's inequality.

(6). Using Cauchy's integral formula, show that $\oint_C \frac{e^{zt}}{z^2+1}\,dz = 2\pi i\,\mathrm{Sin}\,t$, if $t>0$ and C is $|z|=3$.

Solution : The integrand has the singularities given by the roots of $z^2+1=0$ (i.e.,) $z=i$ and $-i$ which are simple (first order). Both lie inside the contour. According to the integral formula, there should be only one factor in the denominator related to singularity. So putting $\frac{1}{z^2+1}$ into partial fractions, we get

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

Then the problem can be written as

$$\oint_C \frac{e^{zt}}{z^2+1}\,dz = \frac{1}{2i}\oint_C \frac{e^{zt}}{z-i}\,dz - \frac{1}{2i}\int_C \frac{e^{zt}}{z+i}\,dz$$

$$= \frac{1}{2i}\left[e^{zt}\right]_{z=i} \times 2\pi i - \frac{1}{2i}\left[e^{zt}\right]_{z=i} \times 2\pi i = \frac{2\pi i}{2i}\left(e^{it} - e^{-it}\right) = 2\pi i\,\mathrm{Sin}t$$

6.12 Summary of the Lesson :

Fundamental definition of a line integral in complex variables is given. Based on the definitiion, some basic integrals are evaluated. The concepts of simply connected region, multiply connected regions and the conversion of multiply connected region into simply connected region are clearly explained.

If $f(z)$ is totaly analytic in the given contour, then Cauchy's (integral) theorem proves that the integral of that function over the closed contour vanishes. Certain of the integrals can be evaluated by simply finding the values of the function at points lying inside the contour and this is given by Cauchy's integral formula. This has been extended to evaluation of integrals in terms of the derivatives of analytic functions. Lastly, converse to the Cauchy's theorem has been proved. Assorted examples have been worked.

6.13 Key Terminology :

Jordan Curve - Contour - Simply connected region - multiply connected region - cross cut - Cauchy's integral theorem - Cauchy's integral formula.

6.14 Reference Books :

1. M.R. Spiegel : "Theory and Problems of Complex Variables" - Schaum Outline Series, Mc-Graw Hill Book Co., 1964

2. B.D. Gupta : "Mathmatical Physics" - Vikas Publishing House Pvt. Ltd., 1980.

3. C.R. Wylie Jr. : "Advanced Engineering Mathematics" - Mc.Graw Hill Book Co.

6.15 Self Assessment Questions :

1. Explain the complex line integral. Starting from the definition find $\int_C z\, dz$, where 'C' is (i) from a to b. and (ii) Closed.

2. Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2| = 4$, (b) the circle $|z-1| = 5$, (c) the square with vertices at $2 \pm 2i, -2 \pm 2i$.

3. Evaluate

(a) $\oint_C \frac{Sin\, \pi z^2 + Cos\, \pi z^2}{(z-1)(z-2)} dz$ (b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ Where 'C' is the Circle $|z| = 3$.

4. Using derivatives theorem on analytic functions, show that $\frac{1}{2\pi i} \int_C \frac{e^{zt}}{(z^2+1)^2} dz = \left(\frac{1}{2} Sin t - t\, Cos\, t\right)$,

if $t > 0$ and 'C' is the circle $|z| = 3$.

5. Apply derivatives theorem on analytic functions to show that $\oint_C \frac{z\, e^{zt}}{(z+1)^3}\, dz = 2\pi i \left(t - \frac{t^2}{2} \right) e^{-t}$, where t > 0 and 'C' is any closed contour enclosing z = -1.

6. What is the value of $\oint_C \frac{z+4}{z^2+2z+5}\, dz$

(a) if C is the circle $|z| = 1$?

(b) If C is the Circle $|z + 1 - i| = 2$?

(c) If C is the circle $|z + 1 + i| = 2$?

Unit – II

Lesson – 7

INFINITE SERIES IN THE COMPLEX PLANE

Objective of the Lesson :

- To expand $f(z)$ in power series in the complex plane.
- To express any analytic function in Taylor series
- To obtain Laurent Series for $f(z)$ analytic in a ring shaped region
- To define various kinds of singularities
- To work out good number of examples for further understanding.

Structure of the lesson :

7.1. Introduction

7.2. Certain concepts in power series

7.3. Taylor's theorem

7.4. Laurent's Theorem

7.5. Classification of singularities

7.6. Residues

7.7. Method of obtaining residues

7.8. Examples

7.9. Summary of the lesson

7.10. Key terminology

7.11. Reference Books

7.12. Self assessment Questions

7.1. Introduction :

Most of the definitions and theorems relating to infinite series of real terms can be applied with little or no change to series whose terms are complex. However, one surprising property of complex analytic function is that they have derivativies of all orders and they can always be represented by power series like Taylor series. But this is not true, in genreral, for real functions. There are real functions which have all orders but cannot be represented bya power series. In many applications, it is necessary to expand functions around points at which or in the neighbourhood of which the functions are not analytic. Obviously, Taylor's series is inapplicable in such cases and a new type of series known as Laurent's expansion is required. Thus the taylor series and Laurent series follow in this lesson.

7.2 Certain Concepts in Power Series :

A series having the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + ... + = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ --------------(1)}$$

is called a power series in z - a. Clearly, the power series (1) converges for z = a. In general, however, the series converges for other points as well. In such case, it can be shown that there exists a positive number R such that (1) converges for $|z-a| < R$ and diverges for $|z-a| > R$, while for $|z-a| = R$, it may or not converge. The region of convergence of the series (i) is given by $|z-a| < R$. where the radius of convergence R is the distance from a to the nearest singularity of $f(z)$. If the nearest singularity of $f(z)$ is at inifinity the radius of convergence is infinite (i.e.,) the series converge for all z.

A series $\sum_{n=1}^{\infty} u_n(z)$ is called absolutely convergent, if the series of absolute values (i.e.) $\sum_{n=1}^{\infty} |u_n(z)|$ converges.

The ratio test for convergence says that if $\underset{n\to\infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right| = L$, then $\sum u_n$ converges (absolutely) if $L<1$ and diverges if $L>1$. If $L=1$, the test fails.

7.3 Taylor's theorem :

If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\underline{|2}}(z-a)^2 + ...$$

Proof : Let z be any point inside C. Construct a circle C_1 with centre at a and enclosing z as shown in Fig. 1. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-z}dt \quad \text{------- (2)}$$

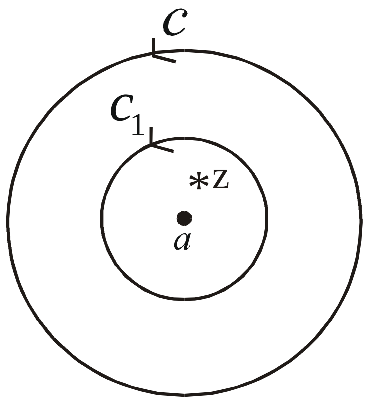


Fig - 1

Now $$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = \frac{1}{t-a}\left[1-\frac{z-a}{t-a}\right]^{-1}$$

$$= \frac{1}{t-a}\left[1+\frac{z-a}{t-a}+\frac{(z-a)^2}{(t-a)^2}+.......\right] \quad \because \left|\frac{z-a}{t-a}\right|<1$$ as t is any point on C and z is inside C_1.

$$= \frac{1}{t-a}\left[1+\frac{z-a}{t-a}+\frac{(z-a)^2}{(t-a)^2}+....+\frac{(z-a)^{n-1}}{(t-a)^{n-1}}+\frac{(z-a)^n}{(t-a)^n}\cdot\frac{1}{1-\frac{z-a}{t-a}}\right]$$

or $$\frac{1}{t-z} = \frac{1}{t-a}+\frac{z-a}{(t-a)^2}+\frac{(z-a)^2}{(t-a)^3}+....+\frac{(z-a)^{n-1}}{(t-a)^n}+\frac{(z-a)^n}{(t-a)^n}\cdot\frac{1}{t-z} \quad \text{---- (3)}$$

using equations (3), equation (2) becomes

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-a}dt+\frac{z-a}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^2}dt+.....+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^n}dt+R_n \quad \text{---(4)}$$

where $R_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{t-a}\right)^n \frac{f(t)}{t-z}\,dt$

Using the derivative formula for analytic function as

$$f^n(a) = \frac{\lfloor n}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^{n+1}}\,dt \quad (\text{ n = 0, 1, ...})$$

Equation (4) becomes

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{\lfloor 2} + ... + \frac{(z-a)^{n-1}.f^{(n-1)}(a)}{\lfloor n-1} + R_n \quad \text{--- (5)}$$

Now we show that $\underset{n\to\infty}{Lt}\, R_n = 0$

$\frac{|z-a|}{|t-a|} = \eta < 1$, where η is a constant.

$f(t)$ is bounded $\quad \therefore |f(t)| \le M \quad$ (constant)

$|t-z| = |(t-a)-(z-a)| \ge |t-a| - |z-a| = r_1 - |z-a|$ where r_1 is the radious of C_1.

$$\therefore |R_n| \le \frac{1}{2\pi}\int_{c_1}\left|\frac{z-a}{t-a}\right|^n . \frac{|f(t)|}{|t-z|}|dt|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\frac{\eta^n.M\,r_1\,d\theta}{r_1-|z-a|} = \frac{\eta^n\,M\,r_1}{r_1-|z-a|} \to 0 \text{ as } n\to\infty \text{ and } \eta<1$$

Hence $f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{\lfloor 2} + ...$ ------ (6)

which is the required result.

The particular case where a = 0 is called Maclaurin series of $f(z)$.

7.4 Laurents's Theorem :

Theorem : If $f(z)$ is analytic inside and on the boundary of the ring shaped region bounded by two concentric circles C_1 and C_2 with centre at a and respective radii $r_1, r_2\,(r_1 > r_2)$, then for all z in the region,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n} \quad \text{--- (7)}$$

where $a_n=\dfrac{1}{2\pi i}\oint_{c_1}\dfrac{f(t)}{(t-a)^{n+1}}dt$ ------- (8)

$$a_{-n}=\frac{1}{2\pi i}\oint_{c_2}\frac{f(t)}{(t-a)^{-n+1}}dt \quad \text{-------- (9)}$$

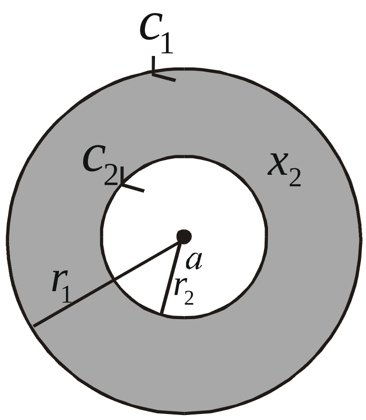


Fig - 2

Proof : Since the given annular region (fig. 2) is a doubly connected region and z is an interior point, then according to Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-z)}dt-\frac{1}{2\pi i}\oint_{C_2}\frac{f(t)}{(t-z)}dt \quad \text{............... (10)}$$

Consider the first integral in (10). We have,

on $C_1,\ |t-a|=r_1,\ t-a=r_1e^{i\theta},\ dt=r_1e^{i\theta}\,i\,d\theta,\ |dt|=r_1\,d\theta \text{ and } |z-a|<|t-a|$

Hence, $\dfrac{1}{t-z}=\dfrac{1}{(t-a)\left[1-\dfrac{z-a}{t-a}\right]}$

$$=\frac{1}{t-a}+\frac{z-a}{(t-a)^2}+.....+\frac{(z-a)^{n-1}}{(t-a)^n}+\frac{(z-a)^n}{(t-a)^{n+1}}.\frac{t-a}{t-z} \quad \text{---- (11)}$$

$$\therefore \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-z}dt=\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-a}dt+\frac{z-a}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^2}dt+......+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^n}dt+R_n$$

$$=a_0+a_1(z-a)+...+a_{n-1}(z-a)^{n-1}+R_n \quad \text{-------- (12)} \quad \text{according to (8)}$$

where $R_n=\frac{1}{2\pi i}\oint_{C_1}\frac{(z-a)^n}{(t-a)^n}\frac{f(t)}{t-z}dt$ -------- (13)

$$|R_n| \le \frac{1}{2\pi}\oint_{C_1}\frac{|z-a|^n}{|t-a|^n}\frac{|f(t)|}{|t-z|}|dt|$$

$$\le \frac{1}{2\pi}\oint_{C_1}\frac{\eta^n M\ r_1\ d\theta}{r_1-|z-a|}$$

i.e., $\le \frac{\eta^n M r_1}{r_1-|z-a|}\to 0$ as $n\to\infty$ and $|\eta|<1$.

(For detailed steps, vide the proof of Taylor's theorem).

So equation (12) becomes

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(t)}{t-z}dt=a_0+a_1(z-a)+a_2(z-a)^2+..... \quad \text{---- (14)}$$

Let us now consider the second integral. We have on $C_2, |t-a|=r_2,\ t-a=r_2 e^{i\theta}$, $dt=r_2 e^{i\theta}\ i\,d\theta$ and $|t-a|<|z-a|$.

Hence, $-\frac{1}{t-z}=\frac{1}{(z-a)\left[1-\frac{t-a}{z-a}\right]}$

$$=\frac{1}{z-a}+\frac{t-a}{(z-a)^2}+....+\frac{(t-a)^{n-1}}{(z-a)^n}+\frac{(t-a)^n}{(z-a)^n}\frac{1}{z-t}$$

so that,

$$-\frac{1}{2\pi i}\oint_{c_2}\frac{f(t)}{t-z}dt=\frac{1}{2\pi i}\oint_{C_2}\frac{f(t)}{z-a}dt+\frac{1}{2\pi i(z-a)^2}\oint_{C_2}(t-a)f(t)dt+.....+\frac{1}{2\pi i}\oint_{C_2}\frac{(t-a)^{n-1}}{(z-a)^n}f(t)dt+S_n$$

$$=\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+....+\frac{a_{-n}}{(z-a)^n}+S_n$$

where $S_n=\frac{1}{2\pi i}\oint_{c_2}\frac{(t-a)^n}{(z-a)^n}\frac{f(t)}{(z-t)}dt$

$\underset{n\to\infty}{Lt}\ S_n=0$ can be proved proceeding on similar lines as in the first integral but the contour is C_2. Thus, combining the reuslts of first and second integrals, Laurent's theorem has been proved.

Note : The part $a_0+a_1(z-a)+a_2(z-a)^2+.....$ is called the analytic part of the Laurent series. The remainder of the series which consists of inverse powers of z - a is called the principal part. If the principal part is zero, the Laurent series reduces to a Taylor series.

The Coefficients of the positive powers of (z - a) in the analytic part, although identical in form with the integrals in Taylor's series, cannot be replaced by the derivative expressions $\frac{f^{(n)}(a)}{\lfloor n}$ since $f(z)$ is not analytic throughout the interior of C_1.

7.5 Classification of Singularities :

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. Various types of singularities exitst. It is possible to classify the singularities of a function $f(z)$ by examination of its Laurent series.

(i) Isolated Singularity : If $z=a$ is a singular point of the function $f(z)$, but if there exists a neighbourhood of a in which there are no other singular points of $f(z)$, then z = a is called an isolated singularity.

(ii) Pole : If the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+....+\frac{a_{-n}}{(z-a)^n} \text{ where } a_{-n}\neq 0$$

then z = a is called a pole of order n.

If n = 1, it is called a simple pole.

If $f(z)$ has a pole at z = a, then $\underset{z\to a}{Lt}\, f(z)=\infty$

For instance,

$$\frac{1}{z(z-1)^2}=\frac{[1+(z-1)]^{-1}}{(z-1)^2}$$

$$=\frac{1}{(z-1)^2}-\frac{1}{(z-1)}+1-(z-1)+..... \qquad 0<|z-1|<1$$

is the Laurent expansion of $\frac{1}{z(z-1)^2}$. Its principal part contains only two terms, namely,

$$\frac{1}{(z-1)^2}-\frac{1}{(z-1)}$$

Hence z = 1 is a pole of second order. This can also be seen from the function as (z-1) is repeated twice in the denominator.

(iii) Essential Singularity : If the principal part contains infinite number of terms of negative powers of z - a, then z = a is called an <u>essential singularity</u> of f(z).

A simple example is $e^{1/z}$ represented by the series $e^{1/z}=1+\frac{1}{z}+\frac{1}{\lfloor 2\, z^2}+\frac{1}{\lfloor 3\, z^3}+......$

which has an essential singularity at z = 0.

(iv) Removable singularity : If a single valued function $f(z)$ is not defined at z = a but $\underset{z\to a}{Lt}\, f(z)$ exists, then z = a is called a <u>removable singularity</u>. In such a case, we define $f(z)$ at z = a as equal to $\underset{z\to a}{Lt}\, f(z)$.

For instance, $f(z)=\frac{\sin z}{z}$, then z = 0 is a removable singularity since $f(0)$ is not defined but $\underset{z\to 0}{Lim}\frac{\sin z}{z}=1$. We define $f(0)=\underset{z\to 0}{Lim}\frac{\sin z}{z}=1$. Note that, in this case

$$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{\lfloor 3}+\frac{z^5}{\lfloor 5}+..........\right) = 1-\frac{z^2}{\lfloor 3}+\frac{z^4}{\lfloor 5}-\frac{z^6}{\lfloor 7}+.....$$

Branch Points : A point $z=z_0$ is called a branch point, a kind of singularity, of the many - valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0. z^a where a is not an integer, log z, $\tan^{-1} z$ are some of the common examples of many valued functions for which z = 0 is a branch point.

vi. Singularity at Infinity :

By letting $z=\frac{1}{t}$ in $f(z)$, we obtain the function $f\left(\frac{1}{t}\right)=F(t)$. Then the nature of singularity at $z=\infty$ is defined to be the same as that of $F(t)$ at t = 0.

A simple example is that $f(z)=z^3$ has a pole of order 3 at $z=\infty$, since $F(t)=f\left(\frac{1}{t}\right)=\frac{1}{t^3}$ has a pole of order 3 at t = 0.

Note : *In many instances, the Laurent expansion of $f(z)$ is found not through the exapansion given in the theorem, but rather by algebric manipulation suggested by the nature of the function. It is often advantageous to express $f(z)$ interms of partial franctions and expand to get appropriate series.*

7.6 Residues :

If $f(z)$ is analytic everywhere in the region bounded by the closed contour C, then by Cauchy's integral theorem.

$$\int_C f(z)\,dz=0 \quad \text{---------} \ (15)$$

If, however, $f(z)$ has a pole at z = a lying inside C, then the integral (15) will, in general, be different from zero. In this case, we may represent $f(z)$ by Laurent series (7). We see that the coefficient a_{-1} as given by equation (9) is

$$a_{-1} = \frac{1}{2\pi i}\int_c f(z)\,dz \quad \text{----- (16)}$$

and therefore $\int_c f(z)\,dz = 2\pi i\, a_{-1}$, the integration being taken in the counterclockwise sense around a simple closed path C which lies in the domain $0<|z-a|<r$.

The coefficient a_{-1} in the development (7) of $f(z)$ is called the <u>residue of</u> $f(z)$ at z = a and we use the notation that

$$a_{-1} = \operatorname{Res}_{z=a} f(z) = \operatorname{Res}(a)$$

It is to be noted that, irrespective of the order of the pole, always the residue is given only by a_{-1} *and not otherwise.*

7.7 Method of obtaining residues :

(i) z = a is a simple pole : Then the Laurent expansion of $f(z)$ has only one term in the pricnipal part and the series is given by

$$f(z) = \left[a_0 + a_1(z-a) + ...\right] + \frac{a_{-1}}{z-a}$$

$$= \phi(z) + \frac{a_{-1}}{z-a}$$

$$\text{or } (z-a)f(z) = (z-a)\phi(z) + a_{-1}$$

Taking Limit as $z \to a$, $\quad \underset{z\to a}{Lt}(z-a)\,f(z) = 0 + a_{-1}$

or $\quad \operatorname{Res}(a) = a_{-1} = \underset{z\to a}{Lt}(z-a)\,f(z)$ ------(17)

for a simple pole.

Note : *Since* $f(z)$ *contains (z - a) as a simple factor in the denominator, (z - a)* $f(z)$ *does not contain (z - a) as it gets cancelled with that in the denominator.*

(ii) z = a is a pole of order 2 : In this case, the principal part contains two terms and the Laurent expansion is

$$f(z)=\phi(z)+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}$$ where $\phi(z)$ is the analytic part.

$$(z-a)^2 f(z)=(z-a)^2\phi(z)+a_{-1}(z-a)+a_{-2}$$ -----(18)

so as to obtian the residue a_{-1}, differentiate (18) with respect to z and then take the limit as $z\to a$.

$$\therefore \underset{z\to a}{Lt}\frac{d\left[(z-a)^2 f(z)\right]}{dz}=\underset{z\to a}{Lt}\frac{d}{dz}\left[(z-a)^2\phi(z)\right]+a_{-1} = 0+a_{-1}$$

or $$a_{-1}=\underset{z\to a}{Lt}\frac{d}{dz}\left[(z-a)^2 f(z)\right]$$ -------- (19)

which is the residue for a second order pole.

(iii) z = a is a pole of order 3 : For a third order pole, Laurent expansion is given by

$$f(z)=\phi(z)+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\frac{a_{-3}}{(z-a)^3}$$

$$(z-a)^3 f(z)=(z-a)^3\phi(z)+a_{-1}(z-a)^2+a_{-2}(z-a)+a_{-3}$$ ------ (20)

To obtain the residue a_{-1}, differentiate (20) with respect to z twice and then take the limit as $z\to a$.

We get

$$\underset{z\to a}{Lt}\frac{d^2}{dz^2}\left[(z-a)^3 f(z)\right]=\underset{z\to a}{Lt}\frac{d^2}{dz^2}\left\{(z-a)^3\phi(z)\right\}+2.1.a_{-1}+0+0$$

$$=0+\underline{|2}\,a_{-1}.$$

$$\therefore a_{-1}=\frac{1}{\underline{|2}}\underset{z\to a}{Lt}\frac{d^2}{dz^2}\left[(z-a)^3 f(z)\right]$$ ------- (21)

(iv) z = a is a pole of mth order : Generalizing the above formulae, we get the residue of $f(z)$ at a pole of mth order as

$$\text{Res}(a)=a_{-1}=\frac{1}{\underline{|m-1}}\underset{z\to a}{Lt}\frac{d^{m-1}\left[(z-a)^m f(z)\right]}{d\,z^{m-1}}$$ ------ (22)

7.8 Examples :

(1). Find the Maclaurin series of $f(z) = \tan z$.

Sol : When $f(z) = \tan z$

$$f'(z) = \sec^2 z = 1 + \tan^2 z = 1 + f^2(z)$$

$$f'(0) = 1$$

observing that $f(0) = 0$, we obtain by successive differentiation

$f'' = 2ff'$ $\quad\quad f''(0) = 0$

$f''' = 2f'^2 + 2ff''$ $\quad\quad f'''(0) = 2 \; or \; \frac{f'''(0)}{\lfloor 3} = \frac{1}{3}$

$f^{(4)} = 6f'f'' + 2ff'''$ $\quad\quad f^{(4)}(0) = 0$

$f^{(5)} = 6f''^2 + 8f'f''' + 2ff^{(4)}$ $\quad\quad f^{(5)}(0) = 16 \; or \; \frac{f^{(5)}(0)}{\lfloor 5} = \frac{2}{15}$

Hence the Maclaurin series is

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots\ldots.. \left(|z| < \frac{\pi}{2}\right)$$

(2). Obtain the Maclaurin series of $f(z) = \frac{1}{\left(1+z^2\right)^2}$

Sol : Let $\phi(z) = \frac{1}{1+z^2}$

$$= 1 - z^2 + z^4 - z^6 + z^8 + \ldots.$$

$$\phi'(z) = \frac{-2z}{\left(1+z^2\right)^2} = -2z + 4z^3 - 6z^5 + 8z^7 - \ldots..$$

or

$$f(z) = \frac{1}{\left(1+z^2\right)^2} = 1 - 2z^2 + 3z^4 - 4z^6 + \ldots..$$

Which is the required Maclaurin series

(3). Find the Taylor series of the function

$$f(z)=\frac{2z^2+9z+5}{z^3+z^2-8z-12}$$ with centre at z = 1.

Sol : Putting the given function into partial fraction, we get

$$f(z)=\frac{1}{(z+2)^2}+\frac{2}{z-3}$$

$$=\frac{1}{\left[3+(z-1)\right]^2}-\frac{2}{2-(z-1)} \quad \because \text{ the centre is at } z = 1.$$

or $$f(z)=\frac{1}{9}\left[\frac{1}{\left(1+\frac{z-1}{3}\right)^2}\right]-\frac{1}{1-\frac{1}{2}(z-1)} \quad \text{------------ (23)}$$

Now consider the expansion

$$\phi(x)=\frac{1}{1+x}=1-x+x^2-x^3+.....$$

$$\phi'(x)=-\frac{1}{(1+x)^2}=-1+2x-3x^2+.....$$

$$\therefore \frac{1}{\left(1+\frac{z-1}{3}\right)^2}=1-2.\frac{z-1}{3}+\frac{3.(z-1)^2}{3^2}-4.\frac{(z-1)^3}{3^3}+....$$

and $$\frac{1}{1-\frac{1}{2}(z-1)}=1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+......$$

$\therefore$ Equation (23) becomes

$$f(z)=\left\{\frac{1}{9}-\frac{2}{9}.\frac{z-1}{3}+\frac{3}{9}\frac{(z-1)^2}{3^2}-\frac{4}{9}\frac{(z-1)^3}{3^3}+.....\right\}-\left\{1-\frac{z-1}{2}-\frac{(z-1)^2}{2^2}-\frac{(z-1)^3}{2^3}......\right\}$$

$$= \frac{-8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 \; .....$$

is the required Taylor expression.

Out of the sngular points 3 and - 2 of $f(z)$ since z = 3 is the nearest to the centre z = 1, the series converges for $|z-1| < 2$.

(4). Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ and determine the region of convergence of the series.

Sol : $f(z) = \sin z$

Let $u = z - \frac{\pi}{4}$ *or* $z = u + \frac{\pi}{4}$. Then we have

$$\sin z = \sin\left(u + \frac{\pi}{4}\right) = \sin u \cos\frac{\pi}{4} + \cos u \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}(\sin u + \cos u)$$

$$= \frac{\sqrt{2}}{2}\left[\left(u - \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}}......\right) + \left(1 - \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}}....\right)\right]$$

$$= \frac{\sqrt{2}}{2}\left[\left(1 + u - \frac{u^2}{\underline{|2}} - \frac{u^3}{\underline{|3}} + \frac{u^4}{\underline{|4}}......\right)\right]$$

$$= \frac{\sqrt{2}}{2}\left(1 + \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{\underline{|2}} - \frac{\left(z - \frac{\pi}{4}\right)^3}{\underline{|3}} + ...\right)$$

since the singularity of sin z nearest to $\frac{\pi}{4}$ is at infinitly, the series converges for all finite values of z. (i.e.) $|z| < \infty$.

(5). Expand $f(z)=\dfrac{z}{(z+1)(z+2)}$ in a Taylor's series a) about z = 0 and b) about z = 2. Determine the region of convergence in each case.

Sol. $f(z)=\dfrac{z}{(z+1)(z+2)}=\dfrac{2}{z+2}-\dfrac{1}{z+1}$

(a) $f(z)=\left(1+\dfrac{z}{2}\right)^{-1}-(1+z)^{-1}$ for $|z|<1$

$$=\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+....\right)-\left(1-z+z^2-z^3.....\right)$$

$=\dfrac{z}{2}-\dfrac{3z^2}{4}+\dfrac{7z^3}{8}......$ is the Taylor expansion around z = 0. The singularities of $f(z)$ are z = -1 and - 2. z = -1 is the nearest singularity to the centre z = 0 at a distance of 1. Hence $|z|<1$ is the region of convergence.

(b). $f(z)=\dfrac{2}{\overline{z-2}+4}-\dfrac{1}{\overline{z-2}+3}$

$$=\frac{1}{2}\left[1+\frac{z-2}{4}\right]^{-1}-\frac{1}{3}\left[1+\frac{z-2}{3}\right]^{-1}$$

$$=\frac{1}{2}\left[1-\frac{z-2}{4}+\frac{(z-2)^2}{4^2}-\frac{(z-2)^3}{4^3}+....\right]-\frac{1}{3}\left[1-\frac{z-2}{3}+\frac{(z-2)^2}{3^2}-\frac{(z-2)^3}{3^3}+.....\right]$$

$=\dfrac{1}{6}-\dfrac{1}{72}(z-2)+\left(\dfrac{1}{32}-\dfrac{1}{27}\right)(z-2)^2....$ is the required expansion.

Out of the singularities -1 and - 2, the nearest singularity to the centre z = 2 is z = -1 at a distance of 3 units. So the radious of convergence is $|z-2|<3$.

(6). Find the Laurent expansion of the function $f(z)=\dfrac{7z-2}{(z+1)z(z-2)}$ in the annulus

(a) $1<|z+1|<3$ (b) $0<|z+1|<1$ (c) $|z+1|>3$

Sol : Applying partial fraction to the given $f(z)$. We get

$$f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2} \text{ --------- } (24)$$

(a) The Laurent expansion is required in the positive and negative powers of z + 1 as the annualr region is $1 < |z+1| < 3$. The first term of equation (24) is already in a negative power of (z + 1). We modify the second and third terms of equation (24) so that z will appear in the combination of z + 1.

$$\frac{1}{z} = \frac{1}{(z+1)-1} = \frac{1}{z+1}\left[1 - \frac{1}{z+1}\right]^{-1} \quad \because |z+1| > 1 \qquad \therefore \frac{1}{|z+1|} < 1 \text{ from the annulus}$$

$$= \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \ldots\right] \text{ and}$$

$$\frac{2}{z-2} = \frac{2}{(z+1)-3} = \frac{-2}{3}\left(1 - \frac{z+1}{3}\right)^{-1} \qquad \because \left|\frac{z+1}{3}\right| < 1 \text{ satisfied by the annulus.}$$

$$= -\frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \ldots\right]$$

$\therefore$ The expansion in the annulur region $1 < |z+1| < 3$ is

$$f(z) = -\frac{3}{z+1} + \left[\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \ldots\right] - \frac{2}{3} - \frac{2}{3^2}(z+1) - \frac{2}{3^3}(z+1)^2 \ldots$$

$$= \left[-\frac{2}{3} - \frac{2}{3^2}(z+1) - \frac{2}{3^3}(z+1)^2 \ldots\ldots\right] + \left[-\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \ldots\ldots\right]$$

(b) In this case also, the expansion is in powers of (z + 1). The first term of equation (24) remaining to be the smae, the second term can be written as the convergent series.

$$\frac{1}{z} = \frac{1}{(z+1)-1} = -\left[1 - \overline{z+1}\right]^{-1} \quad \because |z+1| < 1 \text{ as per the annulur region}$$

$$= -\left[1 + (z+1) + (z+1)^2 + \ldots\right]$$

$$\frac{2}{z-2}=\frac{2}{(z+1)-3}=-\frac{2}{3}\left[1-\frac{z+1}{3}\right]^{-1} \qquad \because\left|\frac{z+1}{3}\right|<1$$

$$=-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+.....\right]$$

So the required expansion in $0<|z+1|<1$ region is

$$f(z)=-\frac{3}{z+1}-\left[1+(z+1)+(z+1)^2+....\right]-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+....\right]$$

$$=\left[-\frac{5}{3}-\left(1+\frac{2}{3^2}\right)(z+1)-\left(1+\frac{2}{3^3}\right)(z+1)^2+......\right]-\frac{3}{z+1}$$

Note : *The given function contains z = -1 as a pole of first order. So the principal part of the Laurent expansion around z = -1 should contain only single term as is true from the expansion obtained.*

Further the residue of $f(z)$ at z = -1 is the coefficient of $\frac{1}{z+1}$ as per definition (i.e.) Res (-1) = -3 which can also be seen to be the same even if we apply the formula as

$$\underset{z\to -1}{Lt}(z+1)\frac{7z-2}{(z+1)z(z-2)}=-3.$$

(c) The convergent expansion in the region $|z+1|>3$ is given by

$$f(z)=-\frac{3}{z+1}+\frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1}+\frac{2}{(z+1)}\left[1-\frac{3}{z+1}\right]^{-1}$$

$$=-\frac{3}{z+1}+\frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+....\right]+\frac{2}{z+1}\left[1+\frac{3}{z+1}+\frac{3}{(z+1)^2}+....\right]$$

$$=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+.....$$

(7). Find the Laurent expansion of $f(z)=\dfrac{\sin z}{\left(z-\frac{\pi}{4}\right)^3}$ with centre at $z=\dfrac{\pi}{4}$.

Sol : $$f(z)=\frac{\sin z}{\left(z-\frac{\pi}{4}\right)^3}=\frac{\sin\left(z-\frac{\pi}{4}+\frac{\pi}{4}\right)}{\left(z-\frac{\pi}{4}\right)^3}$$

$$=\frac{\sin\left(z-\frac{\pi}{4}\right)\cos\frac{\pi}{4}+\cos\left(z-\frac{\pi}{4}\right)\sin\frac{\pi}{4}}{\left(z-\frac{\pi}{4}\right)^3}$$

$$=\frac{1}{\sqrt{2}}\frac{\sin\left(z-\frac{\pi}{4}\right)+\cos\left(z-\frac{\pi}{4}\right)}{\left(z-\frac{\pi}{4}\right)^3}\qquad \text{Put } z-\frac{\pi}{4}=u$$

$$=\frac{1}{\sqrt{2}}\frac{\sin u+\cos u}{u^3}$$

$$=\frac{1}{\sqrt{2}}\frac{1}{u^3}\left[\left(u-\frac{u^3}{\lfloor\underline{3}}+\frac{u^5}{\lfloor\underline{5}}.....\right)+\left(1-\frac{u^2}{\lfloor\underline{2}}+\frac{u^4}{\lfloor\underline{4}}.....\right)\right]$$

$$=\frac{1}{\sqrt{2}}\frac{1}{u^3}\left[1+u-\frac{u^2}{\lfloor\underline{2}}-\frac{u^3}{\lfloor\underline{3}}+\frac{u^4}{\lfloor\underline{4}}+\frac{u^5}{\lfloor\underline{5}}-\frac{u^6}{\lfloor\underline{6}}.....\right]$$

$$=\left[\frac{1}{\sqrt{2}}\frac{1}{u^3}+\frac{1}{\sqrt{2}}\frac{1}{u^2}-\frac{1}{\sqrt{2}\lfloor\underline{2}}\cdot\frac{1}{u}\right]+\left[-\frac{1}{\sqrt{2}\lfloor\underline{3}}+\frac{1}{\sqrt{2}\lfloor\underline{4}}u+\frac{1}{\sqrt{2}\lfloor\underline{5}}u^2.....\right]$$

$$=\frac{1}{\sqrt{2}}\left[\frac{1}{\left(z-\frac{\pi}{4}\right)^3}+\frac{1}{\left(z-\frac{\pi}{4}\right)^2}-\frac{1}{\lfloor\underline{2}}.\frac{1}{\left(z-\frac{\pi}{4}\right)}-\frac{1}{\lfloor\underline{3}}+\frac{z-\frac{\pi}{4}}{\lfloor\underline{4}}+....\right]$$

Note : *Since* $z=\frac{\pi}{4}$ *is a pole of order 3 for* $f(z)$*, the residue of* $f(z)$ *at* $z=\frac{\pi}{4}$ *is given by the coefficient of* $\frac{1}{\left(z-\frac{\pi}{4}\right)}$ *as* $-\frac{1}{\sqrt{2}\,\lfloor 2}$*. This can also be obtained by the formula for residues as*

$$\operatorname{Res}\left(\frac{\pi}{4}\right)=\frac{1}{\lfloor 2}\left\{\frac{d^2}{dz^2}\left[\left(z-\frac{\pi}{4}\right)^3\frac{\sin z}{\left(z-\frac{\pi}{4}\right)^3}\right]_{z=\frac{\pi}{4}}\right\}=\frac{1}{\lfloor 2}\left[\frac{d^2}{dz^2}(\sin z)\right] \text{ at } z=\frac{\pi}{4} \quad = -\frac{1}{\sqrt{2}\,\lfloor 2}$$

(8). Find the residue at the singularity of the function $\frac{\sin z}{z^k}$ by applying the formula and verify by the Laurent expansion.

Sol : $f(z)=\frac{\sin z}{z^k}=\frac{\frac{(\sin z)}{z}}{z^{k-1}}$

Here z = 0 is a pole of order k - 1 as $\frac{\sin z}{z}$ tends to 1 as $z\to 0$ or z = 0 is a removable singularity for $\frac{\sin z}{z}$.

Now $\operatorname{Res}(0)=\underset{z\to 0}{Lt}\frac{1}{\lfloor k-2}\frac{d^{k-2}\left[\frac{\sin z}{z^k}.z^{k-1}\right]}{d\,z^{k-2}}$ ------- (25)

$$=\underset{z\to 0}{Lt}\frac{1}{\lfloor k-2}.\frac{d^{k-2}\left(\frac{\sin z}{z}\right)}{d\,z^{k-2}}$$

$$\frac{\sin z}{z}=\frac{z-\frac{z^3}{\lfloor 3}+\frac{z^5}{\lfloor 5}.......}{z}=1-\frac{z^2}{\lfloor 3}+\frac{z^4}{\lfloor 5}..... \quad \text{-------- (26)}$$

If k is even, k - 2 is even.

$$\therefore (k-2)^{th} \text{ term in equation (26)} = \frac{(-1)^{\frac{k-2}{2}} z^{k-2}}{\lfloor k-2+1}$$

After (k - 2) differentiations, (k - 2)th term becomes independent of z and is given by term independent of $z = \dfrac{(-1)^{\frac{k-2}{2}} . \lfloor k-2}{\lfloor k-2+1} = \dfrac{(-1)^{\frac{k-2}{2}}}{(k-1)}$

So while taking the limit as $z \to 0$

$\operatorname{Res}(0) = \dfrac{1}{\lfloor k-2} . \dfrac{(-1)^{\frac{k-2}{2}}}{(k-1)} = \dfrac{(-1)^{\frac{k-2}{2}}}{\lfloor k-1}$ and the remaining terms vanish.

If k is odd, (k - 2) is odd. Since all the terms in equation (26) are even powers, after (k - 2) differentiations of equation (26), the terms contain powers of z or there is no term independent of z. So in the limit as $z \to 0$, all the terms vanish and hence $\operatorname{Res}(0)$ vanishes.

$$\text{So } \operatorname{Res}(0) = \begin{cases} \dfrac{(-1)^{\frac{k-2}{2}}}{\lfloor k-1} & \text{k is even,} \\ 0 & \text{k is odd.} \end{cases}$$

The Laurent expansion of $\dfrac{\sin z}{z^k}$ is given by the series $\dfrac{\sin z}{z^k} = \dfrac{z - \dfrac{z^3}{\lfloor 3} + \dfrac{2^5}{\lfloor 5}.....}{z^k}$ ----- (27)

If k is even, k - 1 is odd, we pick up the term in equation (27) which gives $\dfrac{1}{z}$ so that its coefficient corresponds to the residue at z = 0.

That term is

$$\frac{(-1)^{\frac{k-2}{2}} \dfrac{z^{k-1}}{\lfloor k-1}}{z^{k-1}.z} = \frac{(-1)^{\frac{k-2}{2}}}{\lfloor k-1} . \frac{1}{z}$$

$$\therefore \operatorname{Res}(0) = \frac{(-1)^{\frac{k-2}{2}}}{\lfloor k-1} \quad \text{(k is even)}$$

If k is odd, the Laurent expansion contains $\frac{1}{z^2}$ term after constant term and not $\frac{1}{z}$ term.

Then $\operatorname{Res}(0)=0$ for odd k.

Hence the result.

(9). Find the resides at the poles of the function $f(z)=\dfrac{e^{zt}}{z^2\left(z^2+2z+2\right)}$

Sol : The poles of the function are given by the roots of $z^2=0$ and $z^2+2z+2=0$

or $z=0$ (second order).

$z=-1+i$ and $-1-i$ (each simple)

$\operatorname{Res}(0)$; $\operatorname{Res}(0) = \underset{z\to 0}{Lt}\ \frac{1}{\lfloor 1}\frac{d}{dz}\left\{\frac{e^{zt}}{z^2\left(z^2+2z+2\right)}z^2\right\}$

$$= \underset{z\to 0}{Lt}\ \frac{\left(z^2+2z+2\right)t\,e^{zt}-e^{zt}\left(2z+2\right)}{\left(z^2+2z+2\right)^2} = \frac{t-1}{2}$$

$\operatorname{Res}(-1+i)$: $\operatorname{Res}(-1+i)= \underset{z\to -1+i}{Lt}\ \frac{\left(z+1-i\right)e^{zt}}{z^2\left(z+1-i\right)\left(z+1+i\right)}$

$$= \underset{z\to -1+i}{Lt}\ \frac{e^{zt}}{z^2\left(z+1+i\right)}$$

$$=\frac{e^{(-1+i)t}}{\left(-1+i\right)^2 2i} = \frac{e^{(-1+i)t}}{4}$$

$\operatorname{Res}(-1-i)$:

$$\operatorname{Res}(-1-i)= \underset{z\to -1-i}{Lt}\ \frac{\left(z+1+i\right)e^{zt}}{z^2\left(z+1+i\right)\left(z+1-i\right)}$$

$$= \underset{z\to -1-i}{Lt}\ \frac{e^{zt}}{z^2\left(z+1-i\right)} = \frac{e^{(-1-i)t}}{\left(1+t\right)^2\left(-2i\right)} = \frac{e^{(-1-i)t}}{4}$$

7.9 Summary of the Lesson :

The necessary basics of the power series is given. The proof of Taylor's theorem for a power series expansion of an analytic function $f(z)$ around any given point of the region is given. A very important Laurent expansion of an analytic function $f(z)$ in an annulur region is given. Various kinds of singularities and the formulae for obtaining residues at the singularities are defined and derived respectively in the light of discussion of the principal part of Laurent expansion. Several variety of examples have been worked for better concepts of singularities and residues.

7.10 Key terminology :

Power Series - Taylor Series - Laurent Series - Singularities - Poles - Branch points - residues

7.11 Reference Books :

1. M.R. Spiegel : 'Theory and Problems of Complex Variables'
   McGraw - Hill Book Co., 1964
2. E. Kreyszig : 'Advanced Engineering Mathematics'
   Wiley Eastern Pvt. Ltd., New Delhi, 1971
3. B.D. Gupta : 'Mathematical Physics'
   'Vikas Publishing House Pvt. Ltd., 1980

7.12 Self Assessment Questions :

1. Find the Taylor series expansion of $f(z)=\tan z$ about $z=\frac{\pi}{4}$.

2. Expand $f(z)=\frac{1}{(1+z)^2}$ around z = -i

3. Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor series about z = 0.

4. Without performing the Taylor expansions for the following functions about the indicated points, write down the region of convergence in each case with reasoning.

   (a) $\frac{z}{e^{z+1}}$ about z = 0 (b) $\frac{z+3}{(z-1)(z-4)}$ about z = 2 (c) $\sec \pi z$ about z = 1.

5. Give the Taylor expnasion of $\frac{z}{z^2+2z+5}$ around z = 1 and determine the radius of convergence.

6. Find the Laurent expansion of $f(z)=\frac{1}{(z-1)(z-2)}$ for the annulur region

   (a) $|z|<1$, (b) $0<|z-1|<1$ (c) $0<|z-2|<1$

7. Expand $f(z)=\frac{1}{z^2(z-i)}$ in the Laurent series in the region (a) $0<|z-i|<1$ and (b) $|z-i|>1$.

8. Find the Laurent series for $\frac{\cos \pi z}{(1-z)^2}$ about the centre z = 1.

9. Find the residues of $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles in the finite plane.

10. Find the residues at the singular points of the function $\frac{1}{(z^4-1)^2}$.

Unit - II

Lesson - 8

Contour Integration of Definite Integrals

Objective of the lesson :

* To prove a theorem on residues
* To apply that theorem for the evaluation of several definite integrals.
* To explain several computation techniques for the evaluation of integrals with examples.
* To make the reader to understand the techniques such as choice of the contour, judging the positions of singularities and applying the residue theorem.
* To give as many worked examples for better concepts and understanding.

Structure of the lesson :

8.1. Introduction :

In the last lesson, Laurent's theorem, particularly the principal part, was studied in detail. Definitions of singularities and the residues at the singularities were given. The formulae for the computation of residues at poles of various orders were derived. These will be used in this lesson to evaluate definite integrals using contours.

8.2. Residue Theorem :

Theorem : If function $f(z)$ is single valued, continuous and analytic within and on the boundary of the closed contour C except at a finite number of singular points $z_1, z_2, ...z_n$ inside C, then

$$\int_c f(z)\,dz = 2\pi i\left[\mathrm{Res}(z_1)+\mathrm{Res}(z_2)+.....+\mathrm{Res}(z_n)\right] \quad ......\ (1)$$

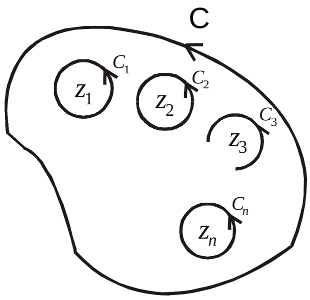


Fig. 1.

Proof : Let us draw a set of circles $C_1, C_2, ..., C_n$ with centres $z_1, z_2, ..., z_n$ with an arbitrarily small radius. The directions (as shown in Fig. 1) of $C_1, C_2, ..., C_n$ and C are all positive when they are taken in anticlock-wise sense. The given function $f(z)$ is every where analytic in the new region (multiply connected) where the outer boundary is C and the inner boundaries are $C_1, C_2, ..., C_n$. Hence by applying Cauchy's integral theorem,

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz .... - \int_{C_n} f(z)\,dz = 0$$

or

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ...+ \int_{C_n} f(z)dz$$

Also

$$\frac{1}{2\pi i}.\int_C f(z)dz = \frac{1}{2\pi i}\int_{C_1} f(z)dz + \frac{1}{2\pi i}\int_{C_2} f(z) + ...+ \frac{1}{2\pi i}\int_{C_n} f(z)dz \quad ....\ (2)$$

But according to the first term in the principal part of Laurent expansion, each term in the RHS of (2) represents the residue $f(z)$ at the respective singularity.

(i.e) $\frac{1}{2\pi i}\int_{C_1} f(z)dz$ = Residue of $f(z)$ at the singularity z_1.

Then Equation (2) represents

$$\frac{1}{2\pi i}\int_C f(z)dz = \text{Res}(z_1) + \text{Res}(z_2) + \ldots\ldots\ldots + \text{Res}(z_n)$$

or

$$\int_c f(z)dz = 2\pi i \sum_{r=1}^{n} \text{Res}(z_r)$$

Hence the theorem is proved.

This important theorem has various applications in connection with complex and real integrals. We see some examples on complex integrals.

Examples :

(1). Integrate $\frac{1}{\left(z^3-1\right)^2}$ in the counterclockwise sense around the circle $C:|z-1|=1$

Solution : The function $f(z)=\frac{1}{\left(z^3-1\right)^2}$ has three poles each of second order and they are given by the roots of $z^3=1$.

So z = 1, $e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$ are the poles (each second order). The distance of each pole from the centre (1, 0) of the given contour $|z-1|=1$ is found as less than the radius 1, then that pole lies inside the contour. Thus the pole z = 1 only lies inside the contour.

So applying Cauchy's theorem for residues, we have

$$\int_c \frac{dz}{\left(z^3-1\right)^2} = 2\pi i \quad \text{[Res (1)]} \ldots\ldots\ldots\ldots (3)$$

Since z = 1 is a pole of second order,

$$\text{Res}(1) = \underset{z\to 1}{Lt} \frac{1}{\lfloor 1} \frac{d}{dz}\left[\frac{1}{\left(z^3-1\right)^2}.\left(z-1\right)^2\right]$$

$$= \underset{z\to 1}{Lt} \frac{d}{dz}\left[\frac{1}{\left(z^2+z+1\right)^2}\right]$$

$$= \underset{z\to 1}{Lt} \frac{-2\left(2z+1\right)}{\left(z^2+z+1\right)^3} = -\frac{2}{9}$$

$\therefore$ Equation (3) becomes $\int_C \frac{dz}{\left(z^3-1\right)^2} = 2\pi i\left(\frac{-2}{9}\right) = \frac{-4\pi i}{9}$

(2) Integrate $\frac{1}{(z-a)^m}$ (m . a +ve integer) in the counterclockwise sense around any simple closed path C enclosing the point z = a.

Solution : z = a is a pole of order 'm' which is given that it lies inside the contour.

Hence $\text{Res}(a) = \underset{z\to a}{Lt} \frac{1}{\lfloor m-1}.\frac{d^{m-1}\left[\frac{(z-a)^m}{(z-a)^m}\right]}{dz^{m-1}} = 0$ (m = 2, 3, ...) But when m = 1, then the pole z = a of the function $\frac{1}{z-a}$ is only simple pole. Then the $\text{Res}(a) = \underset{z\to a}{Lt} (z-a)\frac{1}{z-a}$. Hence the result is

$$\int_C \frac{dz}{(z-a)^m} = \begin{cases} 2\pi i & (m=1) \\ 0 & (m=2,3.....) \end{cases}$$

(3). Integrate $\frac{z}{4z^2-1}$ around C : $|z| = 1$

Solution : The function $f(z) = \frac{z}{4z^2-1} = \frac{z}{(2z-1)(2z+1)}$ has simple poles at $z = \frac{1}{2}, -\frac{1}{2}$ which lie inside the circle.

$$\text{Res}\left(\frac{1}{2}\right) = \underset{z \to \frac{1}{2}}{Lt} \frac{\left(z - \frac{1}{2}\right) z}{(2z-1)(2z+1)} = \frac{1}{2} \cdot \frac{\frac{1}{2}}{1+1} = \frac{1}{8}$$

$$\text{Res}\left(-\frac{1}{2}\right) = \underset{z \to -\frac{1}{2}}{Lt} \frac{\left(z + \frac{1}{2}\right) z}{(2z-1)(2z+1)} = \frac{1}{2} \cdot \frac{-\frac{1}{2}}{-1-1} = \frac{1}{8}$$

$$\therefore \int_C \frac{z}{4z^2 - 1} dz = 2\pi i \left[\text{Res}\left(\frac{1}{2}\right) + \text{Res}\left(-\frac{1}{2}\right) \right]$$

$$= 2\pi i \frac{2}{8} = \frac{\pi i}{2}$$

(4). Evaluate $\int_C \frac{(z+4)^3}{z^4 + 5z^3 + 6z^2} dz$ where C is $|z| = 1$

$$f(z) = \frac{(z+4)^3}{z^4 + 5z^3 + 6z^2} dz = \frac{(z+4)^3}{z^2 (z+3)(z+2)}$$

has the poles z = 0 (2nd order), -3 and -2 each being simple. But z = 0 only lies inside the contour C.

$$\text{Res}(0) = \underset{z \to 0}{Lt} \frac{d}{dz} \left[\frac{(z+4)^3}{z^2 (z+3)(z+2)} \cdot z^2 \right]$$

$$= \underset{z \to 0}{Lt} \frac{d}{dz} \left[\frac{(z+4)^3}{(z+3)(z+2)} \right]$$

$$= \underset{z \to 0}{Lt} \frac{(z^2 + 5z + 6)\left[3(z+4)^2\right] - (z+4)^3 . [2z+5]}{(z^2 + 5z + 6)^2}$$

$$= \frac{6.\ 3.\ 4^2 - 4^3.5}{6^2} = \frac{4^2.2.(-1)}{6^2} = -\frac{8}{9}$$

$$\therefore \int_C \frac{(z+4)^3}{z^4 + 5z^3 + 6z^2} dz = 2\pi i \left(-\frac{8}{9}\right) = -\frac{16\pi i}{9}$$

(5). Find $\int_C \frac{1}{z \sin z} dz$ where C is $|z| = 2$.

Solution : The function $f(z) = \frac{1}{z \sin z}$ has the poles given by z sin z= 0 (i.e.) z = 0 is a second order pole as $z \sin z = z\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} ...\right) = z^2\left(1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} + ...\right)$ and the other simple poles are $z = \pm n\pi$ where n = 1, 2, ... which lie outside the contour.

So, z = 0 (2nd order) only lies inside the contour.

$$\therefore \operatorname{Res}(0) = \underset{z\to 0}{Lt} \frac{d}{dz}\left(\frac{1}{z \sin z} z^2\right)$$

$$= \underset{z\to 0}{Lt} \frac{d}{dz}\left(\frac{z}{\sin z}\right) = \underset{z\to 0}{Lt} \frac{d}{dz}\left(\frac{z}{z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5}}\right) = \underset{z\to 0}{Lt} \frac{d}{dz}\left[\frac{1}{1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5}}\right]$$

$$= \underset{z\to 0}{Lt} - \left(1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} .....\right)^{-2} \cdot \left(-\frac{2z}{\lfloor 3} + \frac{4z^3}{\lfloor 5} ....\right) = 0$$

$$\text{So } \int_C \frac{1}{z \sin z} dz = 0.$$

8.3 Integrals of the type $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$:

We consider the integrals of the type

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where $R(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ finite on the interval $0 \le \theta \le 2\pi$. Setting $e^{i\theta} = z$, we obtain $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$. Then the integrand becomes a rational function of z, say $f(z)$. As θ ranges from 0 to 2π, the variable z ranges around the unit circle $C, |z| = 1$ in the counterclock wise sense. Since $\frac{dz}{d\theta} = i\, e^{i\,\theta}$, we have $d\theta = \frac{dz}{iz}$ so that the given integral takes the form

$$I = \int_C f(z)\frac{dz}{iz}$$

The next step is to find the poles of the integrand and check the poles that lie inside the contour $C:|z| = 1$. Then according to residue the theorem, $I = 2\pi i$ (Sum of the residues of the poles that lie inside the contour).

Examples :

(6). Evaluate $I = \int_0^{2\pi} \frac{\sin^2\theta\ d\theta}{a + b\cos\theta} \quad a > b > 0$

Solution : Choosing the contour C as $|z| = 1$

(i.e.,) $z = e^{i\theta}$., $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$dz = e^{i\theta}\,.\,i\ d\theta$ or $d\theta = \frac{dz}{iz}$

$$\therefore \frac{\sin^2\theta}{a + b\cos\theta} = \frac{-\frac{1}{4}\left(z^2 - 1\right)^2}{z^2\left(a + \frac{b}{2z}\left(z^2 + 1\right)\right)} = -\frac{1}{2z}\frac{\left(z^2 - 1\right)^2}{bz^2 + 2az + b}$$

$$= -\frac{1}{2b}\frac{\left(z^2 - 1\right)^2}{z\left(z^2 + \frac{2a}{b}z + 1\right)}$$

$$\therefore I = -\frac{1}{2bi}\int_C \frac{\left(z^2 - 1\right)^2 dz}{z^2\left(z^2 + \frac{2a}{b}z + 1\right)}$$

The integrand has the poles z = 0 (second order) and $z=\dfrac{-\dfrac{2a}{b}\pm\sqrt{\dfrac{4a^2}{b^2}-4}}{2}=-\dfrac{a}{b}\pm\sqrt{\dfrac{a^2}{b^2}-1}$ each being simple.

Let $\alpha=-\dfrac{a}{b}+\sqrt{\dfrac{a^2}{b^2}-1}$ and $\beta=-\dfrac{a}{b}-\sqrt{\dfrac{a^2}{b^2}-1}$.

$\because \dfrac{a}{b}>1$, then $|\beta|>1$. Since α and β are the roots of $z^2+\dfrac{2a}{b}z+1=0$, then $\alpha\beta=1$. Since $|\beta|>1$, then $|\alpha|=\dfrac{1}{|\beta|}<1$.

So out of the three poles, 0 and α only lie inside the contour.

$$\therefore \operatorname{Res}(0)=\underset{z\to 0}{Lt}\frac{d}{dz}\left[\frac{\left(z^2-1\right)^2.\,z^2}{z^2\left(z^2+\dfrac{2a}{b}z+1\right)}\right]$$

$$=\underset{z\to 0}{Lt}\frac{\left(z^2+\dfrac{2a}{b}z+1\right).4z\left(z^2-1\right)-\left(2z+\dfrac{2a}{b}\right)\left(z^2-1\right)^2}{\left(z^2+\dfrac{2a}{b}z+1\right)^2}=-\frac{2a}{b}$$

$$\operatorname{Res}(\alpha)=\underset{z\to\alpha}{Lt}\frac{(z-\alpha)\left(z^2-1\right)^2}{z^2(z-\alpha)(z-\beta)}=\frac{\left(\alpha^2-1\right)^2}{\alpha^2(\alpha-\beta)}=\frac{\left(\alpha-\dfrac{1}{\alpha}\right)^2}{(\alpha-\beta)}$$

$$=\frac{(\alpha-\beta)^2}{\alpha-\beta}=\alpha-\beta=2\sqrt{\frac{a^2}{b^2}-1}$$

$$\therefore \int_C \frac{\left(z^2-1\right)^2 dz}{z^2\left(z^2+\dfrac{2a}{b}z+1\right)}=2\pi i\left[-\frac{2a}{b}+2\sqrt{\frac{a^2}{b^2}-1}\right]=\frac{4\pi i}{b}\left[-a+\sqrt{a^2-b^2}\right]$$

$$\therefore I=-\frac{1}{2bi}\frac{4\pi i}{b}\left[-a+\sqrt{a^2-b^2}\right]=\frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

(7). Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{\lfloor n}$ where n is a +ve integer.

Solution : Consider the integral

$$\mathrm{I} = \int_0^{2\pi} e^{\cos\theta}\left[\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)\right]d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)}\,d\theta = \int_0^{2\pi} e^{(\cos\theta + i\sin\theta)}\, e^{-in\theta}.d\theta$$

Let the contour be $z = e^{i\theta}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $dz = e^{i\theta}\, i\,d\theta = iz\,d\theta$

$$\therefore \int_0^{2\pi} e^{(\cos\theta + i\sin\theta)}.e^{-in\theta}\,d\theta = \int_c \frac{e^z (z)^{-n}\,dz}{iz} = \frac{1}{i}\int_c \frac{e^z}{z^{n+1}}\,dz$$

The integrand has a pole z = 0 of order n + 1. and this lies inside the countour.

$$\mathrm{Res}(0) = \underset{z\to 0}{Lt} \frac{1}{\lfloor n} \frac{d^n\left(\frac{e^z\, z^{n+1}}{z^{n+1}}\right)}{dz^n} = \underset{z\to 0}{Lt} \frac{1}{\lfloor n} \frac{d^n e^z}{dz^n} = \frac{1}{\lfloor n}$$

$$\therefore I = \frac{1}{\mathrm{i}}\int_C \frac{e^z}{z^{n+1}}dz = \frac{1}{\mathrm{i}}\, 2\pi i\, \frac{1}{\lfloor n} \qquad \text{by residue theorem}$$

$$= \frac{2\pi}{\lfloor n}$$

Now $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\,d\theta = \mathrm{Re.I} = \frac{2\pi}{\lfloor n}$

Hence the result.

(8). Evaluate $I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta}$ $\quad a > 0$

Solution : Let us change the integral limits from 0 to π to 0 to 2π so as to apply full unit circle contour. Then $I = \frac{1}{2}\int_0^{2\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta}$ since the integrand is an even function.

Proceeding as in Ex. (6) for substitution, we have $I = \frac{1}{2}\int_C \frac{a\,dz}{iz\left[a^2 - \frac{1}{4z^2}\left(z^2-1\right)^2\right]}$

$$= \frac{2a}{i}\int_C \frac{z\,dz}{4a^2 z^2 - z^4 + 2z^2 - 1}$$

$$= +2ai \int_C \frac{z\,dz}{z^4 - \left(4a^2+2\right)z^2 + 1}$$

The integrand has the poles given by the roots of $z^4 - \left(4a^2+2\right)z^2 + 1 = 0$

Since it is a quadratic in z^2, let the two roots be $z^2 = \alpha^2$ and $z^2 = \beta^2$ so that $\alpha^2 + \beta^2 = 4a^2 + z$, further, $\alpha^2\beta^2 = 1$ from the biquadratic equation. Let $\left|\alpha^2\right| < 1$, then $\left|\beta^2\right| > 1$.

So $+\alpha$ and $-\alpha$ poles (simple) lie in the contour.

$$\therefore \text{Res}(+\alpha) = \underset{z\to\alpha}{Lt} \frac{(z-\alpha)\,z}{\left(z^2-\alpha^2\right)\left(z^2-\beta^2\right)} = \frac{\alpha}{2\alpha\left(\alpha^2-\beta^2\right)} = \frac{1}{2\left(\alpha^2-\beta^2\right)}$$

Similarly, $\text{Res}(-\alpha) = \frac{1}{2\left[(-\alpha)^2 - \beta^2\right]} = \frac{1}{2\left(\alpha^2-\beta^2\right)}$

$$\therefore I = 2ai\,.\,2\pi i\left[\frac{1}{2\left(\alpha^2-\beta^2\right)} + \frac{1}{2\left(\alpha^2-\beta^2\right)}\right] \quad \text{by residue theorem}$$

$$= \frac{-4\pi a}{\left(\alpha^2-\beta^2\right)} = \frac{4\pi a}{\beta^2-\alpha^2}$$

We know that $\left(\beta^2-\alpha^2\right)^2 = \left(\beta^2+\alpha^2\right)^2 - 4\alpha^2\beta^2 = \left(4a^2+2\right)^2 - 4 = \left(4a^2+4\right)4a^2$

or $\quad \beta^2-\alpha^2 = \sqrt{16a^2\left(a^2+1\right)} = 4a\sqrt{a^2+1}$

$$\therefore I = \frac{4\pi a}{4a\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}$$

(9). Show that $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $a > |b|$ using a suitable contour, Hence deduce the value of $\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2}$

Solution : As in Example (6), if we proceed with the substitution of a unit circle contour $|z|=1$,

$$I=\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\int_c \frac{dz/iz}{a+\frac{b(z^2-1)}{2zi}}=\int_C \frac{2dz}{bz^2+2aiz-b}$$

$$=\frac{2}{b}\int_C \frac{dz}{z^2+\frac{2ai}{b}z-1}$$

The integrand has the poles given by the roots of

$z^2+\frac{2ai}{b}z-1=0$ or $z=\frac{1}{2}\left[-\frac{2ai}{b}\pm\sqrt{\frac{-4a^2}{b^2}+4}\right]=\left[-\frac{ai}{b}\pm\sqrt{-\frac{a^2}{b^2}+1}\right]$

$$=\left[-\frac{ai}{b}\pm i\sqrt{-1+\frac{a^2}{b^2}}\right]$$ since a > b

$$=\frac{-a+\sqrt{a^2-b^2}}{b}i \text{ and } \frac{-a-\sqrt{a^2-b^2}}{b}i$$ which are simple.

If we put $\alpha=\frac{-a+\sqrt{a^2-b^2}}{b}i$ and $\beta=\frac{-a-\sqrt{a^2-b^2}}{b}i$ then $|\alpha|<1$ as $|\beta|>1$ from the given conditions $a>|b|$. and $|\alpha|\,|\beta|=1$.

So the simple pole α only lies inside the contour $|z|=1$.

$$\text{Res}(\alpha)=\underset{z\to\alpha}{Lt}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}=\frac{1}{\alpha-\beta}=\frac{b}{2i\sqrt{a^2-b^2}}$$

$$\therefore I=\frac{2}{b}2\pi i\frac{b}{2i\sqrt{a^2-b^2}}$$ from the residue theorem

$$=\frac{2\pi}{\sqrt{a^2-b^2}}$$ which is the required result.

Now consider $I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = I(a,b) = \frac{2\pi}{\sqrt{a^2-b^2}}$. Taking the partial derivative w.r.t. a on both sides, we get $\int_0^{2\pi} \frac{-1d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}} \cdot -\frac{1}{2} 2\pi$

or $\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$

(10). Evaluate $\int_0^{\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$ by the method of residues.

Solution : Let us change the limits of integration to 0 to 2π as we use full unit circle as the contour.

$$I = \int_0^{\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta \qquad \because \text{ the integrand is even.}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\phi}{5-4\cos\phi} d\phi \qquad \text{if } \pi-\theta = \phi$$

As the contour c is $|z|=1$, $\cos 2\phi = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$

$\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $dz = e^{i\phi}\, i\, d\phi$ or $d\phi = \frac{dz}{iz}$

$$I = \frac{1}{2}\int_C \frac{\frac{1}{2z^2}(z^4+1)}{iz\left[5-\frac{4}{2z}(z^2+1)\right]} = \frac{1}{4i}\int_C \frac{(z^4+1)dz}{z^2(-2z^2+5z-2)}$$

$$= \frac{i}{8}\int_C \frac{(z^4+1)\,dz}{z^2\left(z^2-\frac{5z}{2}+1\right)}$$

$$= \frac{i}{8}\int_C \frac{(z^4+1)dz}{z^2\left(z-\frac{1}{2}\right)(z-2)}$$

The integrand has the poles z = 0 (second order) and $z = \frac{1}{2}$ and 2 as simple poles.

But z = 0 and $\frac{1}{2}$ only lie inside the contour.

$$\operatorname{Res}(0) = \underset{z\to 0}{Lt}\frac{d}{dz}\frac{z^2\left(z^4+1\right)}{z^2\left(z-\frac{1}{2}\right)(z-2)} = \underset{z\to 0}{Lt}\frac{d}{dz}\left(\frac{z^4+1}{z^2-\frac{5z}{2}+1}\right)$$

$$= \underset{z\to 0}{Lt}\frac{d}{dz}\frac{\left(z^2-\frac{5z}{2}+1\right)\left(3z^3\right)-\left(z^4+1\right)\left(2z-\frac{5}{2}\right)}{\left(z^2-\frac{5z}{2}+1\right)^2} = \frac{5}{2}$$

$$\operatorname{Res}\left(\frac{1}{2}\right) = \underset{z\to\frac{1}{2}}{Lt}\frac{\left(z-\frac{1}{2}\right)\left(z^4+1\right)}{z^2\left(z-\frac{1}{2}\right)(z-2)} = \frac{\frac{1}{16}+1}{\frac{1}{4}\left(-\frac{3}{2}\right)} = -\frac{17}{6}$$

$$\therefore I = \frac{i}{8}\,2\pi i\left(\frac{5}{2}-\frac{17}{6}\right)$$ according to residue theorem

$$= \frac{\pi}{12}.$$

8.4 Improper Integrals of rational function : $\int_{-\infty}^{\infty} f(x)\,dx$

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ for which the interval of integration is not finite, is called an improper integral. Such integrals will be evaluated using contour integration techniques.

The method consists of the following steps

(i) Consider the function $f(z)$ by replacing z for x in $f(x)$

(ii) Find the poles of $f(z)$ and their orders.

(iii) Let us choose a semi-circular contour C given by $|z| = R$, Im z > 0 as in Fig. 2. (i.e.) upper half plane. We choose R large enough to include all the poles in the half-plane.

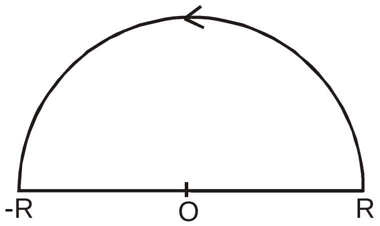


Fig. 2.

(iv) Consider those poles which lie inside the contour and find the residues at these poles only.

(v) Applying Cauchy's residue theorem and taking the limit as $R\to\infty$, we can write

$$\int_C f(z)\,dz = \underset{R\to\infty}{Lt}\int_{C_R} f(z)\,dz + \underset{R\to\infty}{Lt}\int_{-R}^{+R} f(x)\,dx \quad \text{......... (4)}$$

$= 2\pi i$ (sum of residues at those poles which lie inside the contour)

(vi) In most of the problems $\underset{R\to\infty}{Lt}\int_{C_R} f(z)\,dz$ always goes to zero according to the theorem that follows.

Theorem :

Let C_R be an arc of the circle $|z| = R$, having $\theta_1 \le \theta \le \theta_2$ and $R\to\infty$, $z\,f(z)$ tends uniformly to b, then

$$\underset{R\to\infty}{Lt}\int_{C_R} f(z)\,dz = ib(\theta_2 - \theta_1) \quad \text{................ (5)}$$

Proof : By selecting R sufficiently great, we can make

$$|z\,f(z) - b| < \varepsilon \Rightarrow z\,f(z) = b + \eta \quad \text{where } |\eta| < \varepsilon .$$

$$\therefore \int_{C_R} f(z)\,dz = \int_{C_R} \frac{b+\eta}{z}\,dz . \text{ Put } z = \text{Re}^{i\theta} \quad dz = \text{Re}^{i\theta}.i\,d\theta$$

$$\text{So } \int_{C_R} f(z)\,dz = \int_{\theta_1}^{\theta_2} (b+\eta)i\,d\theta = bi(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} \eta i\,d\theta$$

$$\therefore \left|\int_{C_R} f(z)\,dz - bi(\theta_2 - \theta_1)\right| \le \int_{\theta_1}^{\theta_2} |\eta||i\,d\theta| < \varepsilon(\theta_2 - \theta_1)$$

$$\therefore \underset{R\to\infty}{Lt}\int_{C_R} f(z)\,dz = bi\,(\theta_2 - \theta_1)$$

Note : *In all the problems, as the present syllabus is concerned, the value 'b' will be zero.*

Examples :

(11). Using a suitable contour, evaluate $\int_0^\infty \frac{dx}{x^6+1}$

Solution : Consider the function $f(z) = \frac{1}{z^6+1}$. The poles are given y the roots of $z^6+1=0 \Rightarrow z^6=-1$

or $z^6 = e^{(2n+1)\pi i}$ $\therefore z = e^{\frac{2n+1}{6}\pi i}$ (n = 0, 1, 2, 3, 4 and 5)

So the poles are

$$z = e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}, e^{\frac{7\pi i}{6}}, e^{\frac{9\pi i}{6}}, e^{\frac{11\pi i}{6}} = \alpha, \alpha^3, \alpha^5, \alpha^7, \alpha^9, \alpha^{11} \text{ if } \alpha = e^{\frac{\pi i}{6}}.$$

which are all simple.

Let us choose a semi-circular contour C as $|z| = R$, $lm\, z \geq 0$ as shown in Fig. 2. The only poles that lie inside the contour are $\alpha, \alpha^3, \alpha^5$ since their amplitudes are all less than π as can be seen from the fig. 3.

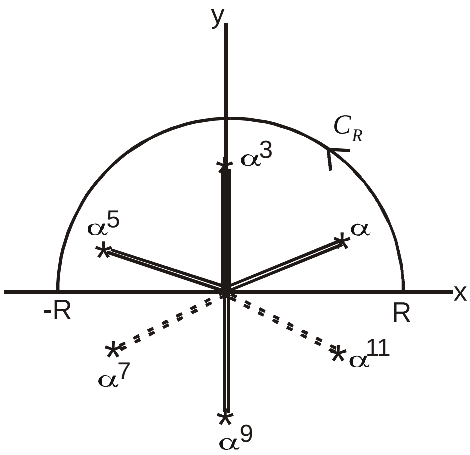


Fig. 3

Now Res (α) = $\underset{z\to\alpha}{Lt} \frac{z-\alpha}{z^6+1}$. If $z=\alpha$ is substituted, we get an indeterminate form as z^6+1 contains $(z-\alpha)$ as a factor. So applying L' Hospital rule for finding the Limit,

$$\left.\begin{array}{l} \operatorname{Res}(\alpha) = \underset{z\to\alpha}{Lt}\frac{1}{6z^5} = \frac{1}{6\alpha^5} = -\frac{\alpha}{6} \\ \text{Similarly } \operatorname{Res}(\alpha^3) = \frac{1}{6(\alpha^3)^5} = \frac{1}{6\alpha^{15}} = \frac{1}{6\alpha^3} \\ \text{and } \operatorname{Res}(\alpha^5) = \frac{1}{6(\alpha^5)^5} = \frac{1}{6\alpha^{25}} = \frac{1}{6\alpha} \end{array}\right\} \quad \because \alpha^6 = -1$$

$$\therefore \text{ Sum of the residues } = -\frac{i}{3}$$

Now applying Cauchy's residue theorem and taking the limit as $R \to \infty$, we get

$$\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz + \underset{R\to\infty}{Lt}\int_{-R}^{+R} f(x)dx = 2\pi i\left(\frac{-i}{3}\right) \quad \text{............ (6)}$$

Now we prove that $\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz = 0$

$$\underset{|z|\to\infty}{Lt}\left|z\,f(z)\right| = \underset{|z|\to\infty}{Lt}\left|\frac{z}{z^6+1}\right| = \underset{|z|\to\infty}{Lt}\frac{|z|}{|z^6+1|}$$

$$= \underset{R\to\infty}{Lt}\, O\left(\frac{1}{R^5}\right) \qquad \text{(order of } \frac{1}{R^5}\text{)}$$

So according to (5) $\underset{R\to\infty}{Lt}\int_{C_R} f(z)\,dz = 0$

Or another method to prove $\int_{C_R} f(z)dz$ to vanish in the limit may be as follows.

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R}\frac{1}{|z^6+1|}|dz|$$

$$\le \left.\int_0^{2\pi}\frac{1}{R^6-1}R\,d\theta\right| \quad \because z = Re^{i\theta},\ dz = Re^{i\theta}\,i\,d\theta,\ |dz| = R\,d\theta \text{ and } |z_1+z_2| > |z_1|-|z_2|$$

i.e., $\le \frac{R}{R^6-1}2\pi$

Thus, in the limit as $R \to \infty$, $\int_{C_R} f(z)dz$ goes to zero. So equation (6) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

or
$$\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

(12). Apply the calculus of residues to prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Solution : Consider $f(z) = \frac{1}{(z^2+1)^3}$. The poles are given by $z = +i$ and $-i$ each being third order.

Let us choose a semi circular contour such that $|z| = R,\ \text{Im}\ (z) \geq 0$ as can be seen in Fig. (3).

$z = +i$ only lies inside the contour.

$$\text{Res}(+i) = \underset{z\to i}{Lt} \frac{1}{\lfloor 2} \cdot \frac{d^2}{dz^2}\left[\frac{1}{(z^2+i)^3}(z-i)^3\right] = \underset{z\to i}{Lt} \frac{1}{\lfloor 2} \frac{d^2}{dz^2} \frac{1}{(z+i)^3}$$

$$= \underset{z\to i}{Lt} \frac{1}{\lfloor 2}(-3)(-4) \cdot \frac{1}{(z+1)^5} = \frac{3}{16i}$$

Applying Residue thoerem and taking the limit as $R \to \infty$,

$$\underset{R\to\infty}{Lt} \int_{C_R} f(z)dz + \underset{R\to\infty}{Lt} \int_{-R}^{R} f(x)dx = 2\pi i \frac{3}{16i} = \frac{3\pi}{8} \quad \text{........... (7)}$$

Now $\left|\int_{C_R} \frac{dz}{(z^2+1)^3}\right| \leq \int_{C_R} \frac{|dz|}{(|z|^2-1)^3}$ $\qquad \because |z_1+z_2| > |z_1| - |z_2|$

$$\leq \int_{C_R} \frac{R\,d\theta}{(R^2-1)^3} \Rightarrow \frac{2\pi R}{(R^2-1)^3} \qquad \because z = \text{R}e^{i\theta},\ |dz| = R\,d\theta$$

So $\underset{R\to\infty}{Lt}\int_{C_R}\frac{dz}{\left(z^2+1\right)^3}=0$

$\therefore$ (7) becomes

$$\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+1\right)^3}=\frac{3\pi}{8}$$

(13). Show that $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=\frac{5\pi}{12}$

Solution : Consider the function $f(z)=\frac{z^2-z+2}{z^4+10z^2+9}$ whose poles are $z=-3i$, $3i, i$ and $-i$ which are simple. Let us choose a semi circular contour $|z|=R, \operatorname{Im}(z)\geq 0$ as given in Fig. 3. The only poles which lie inside the contour are $+3i$ and i .

$$\operatorname{Res}(3i)=\underset{z\to 3i}{Lt}\frac{(z-3i)\left(z^2-z+2\right)}{\left(z^2+1\right)(z+3i)(z-3i)}=\underset{z\to 3i}{Lt}\frac{z^2-z+2}{\left(z^2+1\right)(z+3i)}=\frac{7+3i}{48i}$$

$$\operatorname{Res}(i)=\underset{z\to i}{Lt}\frac{(z-i)\left(z^2-z+2\right)}{(z+i)(z-i)\left(z^2+9\right)}=\underset{z\to i}{Lt}\frac{z^2-z+2}{(z+i)\left(z^2+9\right)}=\frac{1-i}{16i}$$

$$\text{Sum of the residues}=\frac{7+3i}{48i}+\frac{1-i}{16i}=\frac{10}{48i}=\frac{5}{24i}$$

Applying the theorem on residues and tend R to ∞ , we get

$$\underset{R\to\infty}{Lt}\int_{C_R}\frac{z^2-z+2}{z^4+10z^2+9}\,dz+\underset{R\to\infty}{Lt}\int_{-R}^{+R}\frac{x^2-x+2}{x^4+10x^2+9}\,dx=\frac{2\pi i\times 5}{24i}=\frac{5\pi}{12} \quad \text{............. (8)}$$

Now $\left|\int_{C_R}\frac{z^2-z+2}{z^4+10z^2+9}\,dz\right|\leq\int_{C_R}\frac{|z|^2-|z|+2}{|z|^4-10|z|^2-9}|dz|$ as $|z_1+z_2|\leq|z_1|+|z_2|$ and $|z_1+z_2|\geq|z_1|-|z_2|$

$$\leq\int_0^{2\pi}\frac{R^2-R+2}{R^4-10R^2-9}R\,d\theta \quad \text{as } z=\mathrm{R}e^{i\theta},\ |dz|=R\,d\theta$$

$$\Rightarrow 2\pi \frac{R\left(R^2-R+2\right)}{R^4-10R^2-9}$$

$$\therefore \underset{R\to\infty}{Lt} \int_{C_R} \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 0$$

Hence Equation (8) becomes

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{12}$$

8.5 Improper Integrals of rational function when poles lie on the real axis :

The procedure of evaluation involves the following steps :

(i) Choose as usually $f(z)$ by replacing x with z.

(ii) Find the poles and their orders.

(iii) Choose the semi circular contour C with $|z| = R$, $\text{Im}(z) \geq 0$

(iv) Find the poles which lie inside the contour. Further, if there are real poles, they lie on the real axis (i.e.) on the boundary of the contour and they are to be indented by means of a small semi circle such as $|z-a| = \varepsilon$ (arbitrarily small +ve number) as shown in Fig. 4.

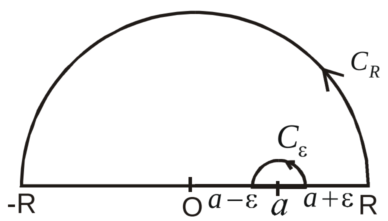


Fig. 4

(v) Find the residues at the poles interior to the contour.

(vi) Applying Cauchy's residue theorem and letting $R\to\infty$ and $\varepsilon\to 0$, we get

$$\underset{R\to\infty}{Lt} \int_{C_R} f(z)dz + \underset{\substack{R\to\infty \\ \varepsilon\to 0}}{Lt} \int_{-R}^{a-\varepsilon} f(x)dx - \underset{\varepsilon\to 0}{Lt} \int_{C_\varepsilon} f(z)dz + \underset{\substack{R\to\infty \\ \varepsilon\to 0}}{Lt} \int_{a+\varepsilon}^{R} f(x)dx$$

$= 2\pi i$ (sum of the residues of interior poles) .... (9)

Having known the evaluation of $\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz$, the integral $\underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon} f(z)dz$ can be evaluated from the following theorem.

Theorem : If $\underset{z\to a}{Lt}(z-a)f(z)=b$, where 'b' is a constant, and if C_ε is an arc $\theta_1 \le \theta \le \theta_2$ of circle $|z-a|=\varepsilon,$ then $\underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon} f(z)dz = ib(\theta_2-\theta_1)$ .... (10)

Proof : For a given δ, we can find ρ such that $|(z-a)f(z)-b|<\delta$ for all $|z-a|<\rho$. Again selecting $\varepsilon<\rho$, we shall have $|(z-a)f(z)-b|<\delta$ on the arc C_ε.

$$\therefore (z-a)f(z) = b+\eta \text{ where } |\eta|<\delta$$

$$\therefore \int_{C_\varepsilon} F(z)dz = \int_{C_\varepsilon} \frac{b+\eta}{z-a}dz \text{, Now put } z-a=\varepsilon e^{i\theta} \therefore dz = \varepsilon e^{i\theta}\, i\, d\theta \text{ (or) } \frac{dz}{z-a} = i\,d\theta$$

$$\therefore \int_{C_\varepsilon} f(z)dz = \int_{\theta_1}^{\theta_2} bi\ d\theta + \int_{\theta_1}^{\theta_2} \eta i\ d\theta$$

or $$\left|\int_{C_\varepsilon} f(z)dz - bi(\theta_2-\theta_1)\right| \le \int_{\theta_1}^{\theta_2} |\eta|\,|i|\,|d\theta| < \delta(\theta_2-\theta_1)$$

$$\therefore \underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon} f(z)dz = i(\theta_2-\theta_1)b$$

Examples

(14). Choosing a suitable contour, evaluate $\int_0^\infty \frac{dx}{1-x^4}$.

Solution : Consider the function $f(z)=\frac{1}{1-z^4}$ and the poles are given by the roots of $1-z^4=0$ (i.e.) z = +1, -1, +i, and -i, all being simple. Let us choose a semicircular contour C such that $|z|=R$, $\text{Im}(z)\ge 0$. The only pole which lies inside the contour is +i and poles +1 and -1 lie on the real axis and hence they are to be indented as shown in Fig. 5.

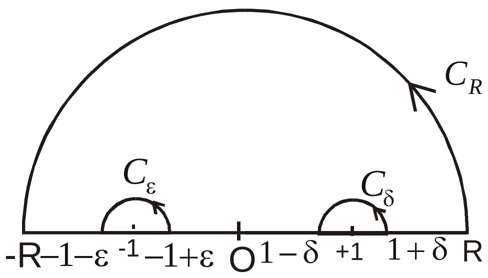


Fig. 5

$$\text{Res}(i)\ \underset{z\to i}{Lt}\frac{(z-i)}{(1-z^2)(z+i)(z-i)}=\frac{1}{4i}$$

Now applying Residue theorem, and taking the limits as $R\to\infty$, ε and δ tending to zero, we get

$$\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz+\underset{\substack{R\to\infty\\ \varepsilon\to 0}}{Lt}\int_{-R}^{-1-\varepsilon} f(x)dx-\underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon} f(z)dz+\underset{\substack{\varepsilon\to 0\\ \delta\to 0}}{Lt}\int_{-1+\varepsilon}^{1-\delta} f(x)dx$$

$$-\underset{\delta\to 0}{Lt}\int_{C_\delta} f(z)dz+\underset{\substack{\delta\to 0\\ R\to\infty}}{Lt}\int_{1+\delta}^{R} f(x)dx\ =2\pi i\ \text{Res}(+i)\ ....\ (11)$$

Integral around C_R :

$$\left|\int_{C_R}\frac{1}{1-z^4}dz\right|\le\int_{C_R}\frac{|dz|}{\left|1-z^4\right|}\le\int_0^{2\pi}\frac{R\,d\theta}{R^4-1}\qquad z=Re^{i\theta},\quad dz=\text{R}e^{i\theta}.\,i\,d\theta,\quad |dz|=\text{R}\,d^{i\theta}\ i\,d\theta$$

$$\Rightarrow\le\frac{2\pi R}{R^4-1}\longrightarrow 0\quad\text{as } R\to\infty$$

$$\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz=0$$

Integral around C_ε :

$$\underset{z\to -1}{Lt}(z+1)\frac{1}{1-z^4}=\underset{z\to -1}{Lt}\frac{z+1}{(1+z)(1-z)(1+z^2)}=\frac{1}{4}$$

$\therefore$ According to Equation (10) of the theorem,

$$\underset{\varepsilon \to 0}{Lt} \int_{C_\varepsilon} \frac{1}{1-z^4}\, dz = i.\frac{1}{4}\left(\pi - 0\right) = \frac{\pi i}{4}$$

Integral around C_δ :

$$\underset{z \to 1}{Lt} \frac{\left(z-1\right)}{\left(1+z\right)\left(1-z\right)\left(1+z^2\right)} = \frac{-1}{4}$$

$$\therefore \underset{\delta \to 0}{Lt} \int_{C_\delta} \frac{1}{1-z^4}\, dz = i\left(\frac{-1}{4}\right)\left(\pi - 0\right) = -\frac{\pi i}{4}$$

$\therefore$ Equation (11) becomes

$$0 + \int_{-\infty}^{-1} f\left(x\right)dx - \frac{\pi i}{4} + \int_{-1}^{1} f\left(x\right)dx + \frac{\pi i}{4} + \int_{1}^{\infty} f\left(x\right)dx = 2\pi i\, \frac{1}{4i}$$

or $$\int_{-\infty}^{\infty} \frac{dx}{1-x^4} = \frac{\pi}{2}$$

or $$\int_{0}^{\infty} \frac{dx}{1-x^4} = \frac{\pi}{4}$$

(15). Use the method of contour integration to prove that

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}\, dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{\pi a}{2}\right) \quad 0 < a < 2$$

Solution : Consider the function $f\left(z\right) = \dfrac{z^{a-1}}{1+z^2}$ which has the singularities.

$z = 0$ (branch point) as z^{a-1} is a many valued function for $0 < a < 2$

and $z = \pm i$ (each being simple pole)

Choose a semi circular contour C such that $|z| = R$, $\operatorname{Im}(z) \geq 0$

z = 0 singularity lies on the real axis and hence it is to be indented as noted in the Fig. 6

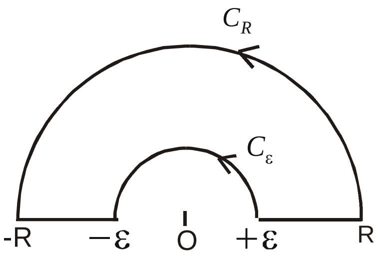


Fig. 6

$z = +i$ only is the pole lying inside the contour.

$$\therefore \operatorname{Res}(i) = \underset{z\to i}{Lt}\ \frac{(z-i)z^{a-1}}{(z+i)(z-i)} = \frac{i^{a-1}}{2i} = -\frac{1}{2}i^a = -\frac{e^{\frac{ia\pi}{2}}}{2}$$

Applying residue theorem and taking the limits as $R\to\infty,\ \varepsilon\to 0$

$$\underset{R\to\infty}{Lt}\int_{C_R} f(z)dz + \underset{\substack{R\to\infty\\ \varepsilon\to 0}}{Lt}\int_{-R}^{-\varepsilon} f(x)dx - \underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon} f(z)dz + \underset{\substack{\varepsilon\to 0\\ R\to\infty}}{Lt}\int_{\varepsilon}^{R} f(x)dx = 2\pi i\times\left(\frac{-1}{2}\right)e^{ia\frac{\pi}{2}} \quad \text{......... (12)}$$

Now $\left|\int_{C_R} f(z)dz\right| \le \int_{C_R}\frac{|z|^{a-1}}{|1+z^2|}|dz| \qquad z = \mathrm{R}\,e^{i\theta};\ |dz| = Rd\theta,\ |1+z^2| > |z|^2 - 1$

$$\le \frac{2\pi R\ R^{a-1}}{R^2-1}$$

(i.e.) $\le \dfrac{2\pi\ R^a}{R^2-1} \to 0$ as $R\to\infty$ for $0<a<2$

$$\therefore \underset{R\to\infty}{Lt}\int_{C_R} f(z)dz = 0$$

$$\underset{\varepsilon\to 0}{Lt}\int_{C_\varepsilon}\frac{z^{a-1}}{1+z^2}dz = i(\pi-0).\ \underset{z\to 0}{Lt}\ \frac{z\ z^{a-1}}{1+z^2} = 0$$

$\therefore$ Equation (12) becomes

$$\int_{-\infty}^{0}\frac{x^{a-1}}{1+x^2}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1+x^2}dx = -\pi i\, e^{\frac{\pi a i}{2}}$$

or $\int_0^{\infty} \frac{(-1)^{a-1}\, t^{a-1}}{1+t^2}\, dt + \int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i\, e^{\frac{\pi a i}{2}}$

(i.e.) $-e^{\pi a i} \int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx + \int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i\, e^{\frac{\pi a i}{2}}$

or $\int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{-\pi i\, e^{\frac{\pi a i}{2}}}{1-e^{\pi a i}} = -\frac{\pi}{2} \frac{2i}{e^{-\frac{\pi a i}{2}} - e^{\frac{\pi a}{2} i}}$

$$= \frac{\pi}{2} co\sec \frac{\pi a}{2}$$

8.6 Integrals of the type $\int_{-\infty}^{\infty} f(x) \cos mx\, dx$ and $\int_{-\infty}^{\infty} f(x) \sin mx\, dx$:

When $f(x)$ is a rational function, such types of integrals occur in connection with Fourier integrals.

The evaluation of those integrals by counter integration methods involves the following steps.

(i) Consider the function $\phi(z) = f(z) e^{imz}$

(ii) Find the poles and any other singularities such as branch points

(iii) Choose the contour C such that $|z| = R$, $\text{Im}(z) \geq 0$

(iv) If there are singular points lying on the real axis, let them be indented. If there are poles lying inside the contour, find the residues.

(v) Apply Resiude theorem and take the limits.

While proving the integral on major semi circular contour to be zero in the $Lt\ R \to \infty$, equation (5) or usual procedure which has been adopting is not useful. However, Jordan's lemma is useful.

Jordan's inequality :

When $0 \leq \theta \leq \frac{\pi}{2}$, we have $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$

i.e., $\frac{2\theta}{\pi} \leq \sin\theta \leq \theta$ .... (13)

Which is caled the Jordan's Inequality.

Jordan's Lemma :

If $f(z) \to 0$ uniformly as $z \to \infty$, then $\underset{R\to\infty}{Lt} \int_{C_R} e^{imz} f(z)dz = 0$ m > 0 ------- (14)

Where C_R denotes the semi circle $|z| = R, \operatorname{Im}(z) > 0$.

Proof : Here R is large enough to include all the singularities within C_R and none on its boundary.

Since $\underset{R\to\infty}{Lt} f(z) = 0$, it follows that for all the points on C_R, $|f(z)| < \varepsilon$, ε being a small +ve number. Now $\left|\int_{C_R} e^{imz} f(z)dz\right|$

$$\le \int_{C_R} \left|e^{imz}\right| |f(z)| |dz| \le \varepsilon \int_{C_R} \left|e^{imz}\right| |dz|$$

$$\le \varepsilon \int_0^{\pi} \left|e^{im(R\cos\theta + iR\sin\theta)}\right| R\, d\theta \qquad \text{Putting } z = \mathrm{Re}^{i\theta},\ |dz| = R\, d\theta$$

(i.e.) $\le \varepsilon \int_0^{\pi} e^{-mR\,Sin\theta} R\, d\theta$

(i.e.,) $\le 2\varepsilon R \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$ Where Jordan's inequality can be applied.

$$\le 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-mR\frac{2\theta}{\pi}}\, d\theta \qquad \because \sin\theta \ge \frac{2\theta}{\pi},\ e^{-\sin\theta} \le e^{-\frac{2\theta}{\pi}}$$

i.e., $\le \frac{e\pi}{m}\left(1 - e^{-mR}\right) \to \theta$ as $R \to \infty$

$$\underset{R\to\infty}{Lt} \int_{C_R} e^{imz} f(z)dz = 0$$

Example

(16). Prove by contour integration that

$$\int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

Solution : Consider the function $\phi(z) = \frac{e^{imz}}{z}$ where $f(z)$ is taken to be $\frac{1}{z}$ which has z = 0 as a simple pole.

Let us choose the semi circular contour 'C' such that $|z| = R$, $Im(z) \geq 0$ wherein z = 0 is to be indented as shown in Fig. 7.

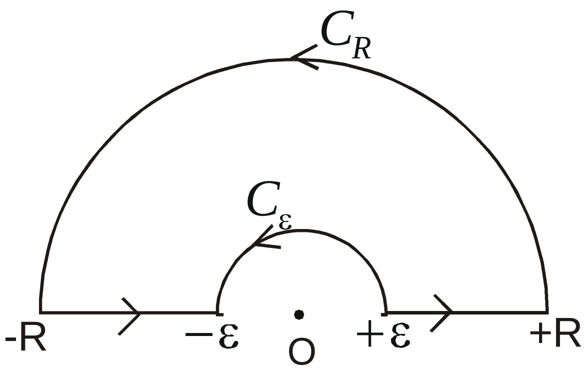


Fig. 7

There are no poles lying inside.

Now applying Cauchy's residue theorem and taking the limits as $R \to \infty$ and $\varepsilon \to 0$, we get

$$\underset{R\to\infty}{Lt} \int_{C_R} \frac{e^{imz}}{z} dz + \underset{\substack{R\to\infty \\ \varepsilon\to 0}}{Lt} \int_{-R}^{-\varepsilon} \frac{e^{imx}}{x} dx - \underset{\varepsilon\to 0}{Lt} \int_{C_\varepsilon} \frac{e^{imz}}{z} dz + \underset{\substack{\varepsilon\to 0 \\ R\to\infty}}{Lt} \int_{\varepsilon}^{R} \frac{e^{imx}}{x} dx = 0 \quad \text{........ (15)}$$

Now $$\left| \int_{C_R} \frac{e^{imz}}{z} dz \right| \leq \int_{C_R} \frac{|e^{imz}|}{|z|} |dz| \qquad C_R : z = Re^{i\theta}, \ |dz| = R\, d\theta$$

$$\leq \int_0^\pi \frac{e^{-mR\sin\theta}}{R} R\, d\theta$$

$$\leq 2 \int_0^{\pi/2} e^{\frac{-2mR\theta}{\pi}} d\theta \qquad \text{by Jordan's inequality.}$$

$$\leq \frac{m\pi}{R}\left(1 - e^{-mR}\right) \to 0 \text{ as } R \to \infty$$

$$\therefore \underset{R\to\infty}{Lt} \int_{C_R} \frac{e^{imz}}{z} dz = 0$$

Again $\underset{\varepsilon \to 0}{Lt} \int_{C_\varepsilon} \frac{e^{imz}}{z} dz = i(\pi - 0)b$ where $b = \underset{z \to 0}{Lt} \frac{(z-0)e^{imz}}{z} = 1$

$$= \pi i$$

$\therefore$ Equation (15) becomes

$$\int_{-\infty}^{0} \frac{e^{imx}}{x} dx - \pi i + \int_{0}^{\infty} \frac{e^{imx}}{x} dx = 0$$

or $$-\int_{0}^{\infty} \frac{e^{-imx}}{x} dx + \int_{0}^{\infty} \frac{e^{imx}}{x} dx = \pi i$$

$$2i \int_{0}^{\infty} \frac{\sin mx}{x} dx = \pi i$$

or $$\int_{0}^{\infty} \frac{\sin mx}{x} = \frac{\pi}{2}$$

(17). By contour integration method evaluate $\int_{0}^{\infty} \frac{1 - \cos x}{x^2} dx$

Solution : Consider $\phi(z) = \frac{1-e^{iz}}{z^2} = \frac{(1-e^{iz})/z}{z}$

in which z = 0 is a simple pole (since z = 0 as per the numerator is a removable singularity). Choose the semi circular contour C. $|z| = R$, $lm(z) \geq 0$. Since z = 0 pole lies on the real axis and it is to be indented as shown in Fig. 7. There are no poles lying inside. Now applying Cauchy's residue theorem and letting R tend to infinity and ε to zero, we have

$$\underset{R\to\infty}{Lt} \int_{C_R} \frac{1-e^{iz}}{z^2} dz + \underset{\substack{R\to\infty \\ \varepsilon\to 0}}{Lt} \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx - \underset{\varepsilon\to 0}{Lt} \int_{C_\varepsilon} \frac{1-e^{iz}}{z^2} dz + \underset{\substack{\varepsilon\to 0 \\ R\to\infty}}{Lt} \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2} dx = 0 \quad \text{......... (16)}$$

Now $\left| \int_{C_R} \frac{1-e^{iz}}{z^2} dz \right| \leq \int_{C_R} \frac{|1-e^{iz}|}{|z|^2} |dz|$ Put $z = Re^{i\theta}$, $|dz| = R\, d\theta$

$$\leq \int_{0}^{\pi} \frac{\left(1+e^{-R\sin\theta}\right) R\ d\theta}{R^2} \to 0$$ as $R \to \infty$ by applying Jordan's lemma

$$\underset{\varepsilon \to 0}{Lt} \int_{C_\varepsilon} \frac{1-e^{iz}}{z^2} dz = i(\pi - 0)b \qquad \text{where } b = \underset{z \to 0}{Lt}\ z\ \frac{1-e^{iz}}{z^2} = -i \text{ (by L Hospital's rule)}$$

$$= \pi$$

$\therefore$ Equation (16) becomes $$\int_{-\infty}^{0} \frac{1-e^{+ix}}{x^2} dx + \int_{0}^{\infty} \frac{1-e^{ix}}{x^2} dx = \pi$$

or $$\int_{0}^{\infty} \frac{1-e^{-ix}}{x^2} dx + \int_{0}^{\infty} \frac{1-e^{ix}}{x^2} dx = \pi$$

(i.e.) $$\int_{0}^{\infty} \frac{2-2\cos x}{x^2} dx = \pi$$

$$\therefore \int_{0}^{\infty} \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}$$

8.7 Evaluation of Integrals using double circular or full circular contours :

Let C_R denote the circle $|z| = R$ where $R \to \infty$ and C_ε the circle $|z| = \varepsilon$ where $\varepsilon \to 0$ these two circles are joined along a cross cut which includes the positive side of the real axis. This type of contour is usually used in evaluating integrals involving many valued functions having branch point at z = 0. Further, poles of $f(z)$ lying only on the positive side of the real axis are to be indented. In other words, the poles lying on the negative side of the real axis are considered to be interior poles where the residues are to be computed.

Note : *There is no special merit in using a particular curve as contour for a particular integration, but infact any of the semicircle, circle, quadrant of a circle or a rectangle whichever is suitable can be used as a contour unless otherwise stated.*

Examples

(18). Using a full circular contour, prove that $\int_{0}^{\infty} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin a\pi} \quad 0 < a < 1$

Solution : Consider the function $f(z) = \frac{z^{a-1}}{1+z} \quad 0 < a < 1$. This has a simple at $z = -1$ and z = 0 is a singularity called branch point as it is given by z^{a-1} $0 < a < 1$, the many valued function. Now let us choose a full circular contour $|z| = R$ having z = 0 indented by a full circle $|z| = \varepsilon$ with a cross cut as shown in Fig. 8.

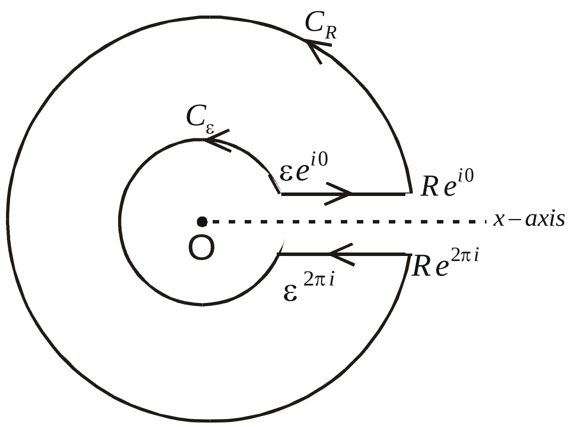


Fig. 8

Residue is to be obtained at z = -1 which is an interior point.

$$\operatorname{Res}(-1) = \underset{z\to -1}{Lt} \frac{(z+1)z^{a-1}}{1+z} = (-1)^{a-1} = e^{\pi i(a-1)} \quad \because 0<a<1$$

Hence by Cauchy's residue theorem, applying limits as $R\to\infty$, $\varepsilon\to 0$, we have

$$\underset{R\to\infty}{Lt} \int_{C_R} \frac{z^{a-1}}{1+z} dz + \underset{\substack{R\to\infty\\ \varepsilon\to 0}}{Lt} \int_{Re^{2\pi i}}^{\varepsilon e^{2\pi i}} \frac{\left(xe^{2\pi i}\right)^{a-1}}{\left(1+x\,e^{2\pi i}\right)} d\left(xe^{2\pi i}\right) - \underset{\varepsilon\to 0}{Lt} \int_{C_\varepsilon} \frac{z^{a-1}}{1+z} dz + \underset{\substack{\varepsilon\to 0\\ R\to\infty}}{Lt} \int_{\varepsilon e^{i0}}^{\mathrm{Re}^{i0}} \frac{\left(xe^{i0}\right)^{a-1}}{1+xe^{i0}} d\left(xe^{i0}\right).$$

$$= 2\pi i\left[\operatorname{Res}(-1)\right] \quad \text{.......... (17)}$$

$$\left|\int_{C_R} \frac{z^{a-1}}{1+z} dz\right| \le \int_0^{2\pi} \frac{R^{a-1}}{R-1} R\, d\theta \qquad \because z = \mathrm{Re}^{i\theta}, |dz| = Rd\theta$$

i.e. $\le 2\pi \dfrac{R^a}{R-1} \to 0$ as $R\to\infty$ as $0<a<1$

So $\underset{R\to\infty}{Lt} \int_{C_R} \frac{z^{a-1}}{1+z} dz = 0$

Similarly, $\underset{\varepsilon\to 0}{Lt} \int_{C_\varepsilon} \frac{z^{a-1}}{1+z} dz = i(2\pi - 0)b$ where $b = \underset{z\to 0}{Lt} \frac{(z-0)z^{a-1}}{1+z} = 0$

Now $\underset{\substack{R\to\infty \\ \varepsilon\to 0}}{Lt} \int_{Re^{2\pi i}}^{\varepsilon e^{2\pi i}} \frac{\left(xe^{2\pi i}\right)^{a-1}}{\left(1+x\,e^{2\pi i}\right)} d\left(xe^{2\pi i}\right) = \int_{\infty}^{0} \frac{x^{a-1}\; e^{2\pi i(a-1)}}{1+x} dx$

$$= -\int_0^\infty \frac{x^{a-1}}{1+x} e^{2\pi i a}\, dx$$

$\therefore$ Equation (17) can be written as

$$0 - e^{2\pi i a} \int_0^\infty \frac{x^{a-1}}{1+x} dx - 0 + \int_0^\infty \frac{x^{a-1}}{1+x} dx = -2\pi i\, e^{\pi a i}$$

or $$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \frac{-2\pi i\; e^{\pi a i}}{1-e^{2\pi i a}} = \pi \frac{2i}{e^{\pi a i} - e^{-\pi a i}} = \frac{\pi}{\sin a\pi}$$

8.8 Summary :

This entire lesson is concentrated on evolving techniques for the evaluation of various types of integrals using differnt contours. The integrals using different contours. The integrands, possessing many kinds of singularities, are useful in the required integrals. Such integrands occur as functions in stability problem, statistical mechanics, optics etc.,

The choice of a suitable contour depends on the problem on hand. One must be careful about the many valued functions wherein branch points occur as singularities. In all the problems we come across, the integral over the major contour as $R\to\infty$ goes to zero. Care must be excercised in judging the poles lying inside, outside and on the real axis (for semi circular contours) or on the positive side of the real axis (for full circular contours) and in applying the theorem on residues.

8.9 Key Terminology :

Residues - Unit Circular Contour - Indentation - Jordan's inequality - Branch points - Full circular contours.

8.10 Reference Books :

1. M.R. Spiegel - 'Theory and problems of complex variables' Mc GrawHill Book Co. 1964
2. B.D. Gupta ' - Mathematical Physics' Vikas Publishing House Pvt. Ltd., 1980
3. P.P. Gupta, R.P.S. Yadav and G.S. Malik - 'Mathemerical Physics' Kedarnath Ramnath, Meerut, 1980.

8.11 Self Assessment Questions :

1. If C is the circle $|z| = 2$, evaluate $\int_C \frac{z}{\cos z} dz$

2. Find $\int_C \frac{Cot\, z}{z} dz$ where $|z| = 1$

3. Show that $\int_0^{\pi} \frac{\cos 2\theta\ d\theta}{1+k^2-2k\cos\theta}$ is equal to (i) $\frac{\pi k^2}{1-k^2}$ $\left(k^2<1\right)$ (ii) $\frac{\pi}{k^2\left(k^2-1\right)}$ $\left(k^2>1\right)$

Hint : Proceed as in Example (10)

4. Choosing a suitable contour, prove that $\int_0^{2\pi} \frac{Cos\, 2\theta}{5+4Cos\,\theta} d\theta = \frac{\pi}{6}$

5. Show that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\cos\theta\ d\theta = \pi$

6. By the method of residues, evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$ $-1< p<1$.

7. Using contour integration techniques, show that $\int_0^{\infty} \frac{dx}{x^4+x^2+1} = \frac{\pi\sqrt{3}}{6}$

8. Evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$ by the method of residues.

9. By contour integration, evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x+1)\left(x^2+2\right)}$

10. With a proper choice of the contour, prove that $\int_0^{\infty} \frac{x^{a-1}}{1-x} dx = \pi\cot a\pi$ $0<a<1$

11. Show that $\int_0^{\infty} \frac{\cos mx}{x^2+1} dx = \frac{\pi}{2} e^{-m}$ $m>0$

12. By theory of residues, show that $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

13. Evaluate $\int_0^\infty \frac{\cos x}{x^2+a^2}dx$ by contour integration.

14. Show that, if $a \geq b \geq 0$, then $\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b-a)$

15. Evaluate $\int_0^\infty \frac{\cos ax}{(x^2+a^2)(x^2+b^2)} dx \quad a>0, b>0$

Unit - III

Lesson - 9

CARTESIAN TENSORS

Objective of the lesson :

> To develop the concept of a tensor by defining scalars and vectors
> To distinguish between 'General tensors' and 'Cartesian tensors'
> To explain tensor notation and summation convention
> To confine to 'Cartesian tensors' in this lesson and explain tensors of several ranks.
> To give several properties of tensors
> To treat special cartesian invariant tensors.
> To explain stress and strain tensors along with their physical significance.

Structure of the Lesson :

9.1 Introduction :

The simplest quantities of Physical interest are those having only one component independent of the coordniate system used. If we make coordniate transformations without changes of fundamental units, these quantities remain invaraint. Such quantities are called <u>scalars</u> or <u>invariants</u> whose magnitudes are the functional values.

Next are the physical quantities which for their complete specification, require, as many components as the space has dimensions. Such quantities are 'n' components in an 'n' dimensional space. In a particular coordniate system, the magnitudes of these components are fixed. When the coordniate transformation takes place, the magnitudes of the components change from system to system. If we wish to call that physical quantity as 'Vector' which represents the magnitude and direction of that quantity, its value (magnitude) should be the same in every system. Any changes in the magnitudes of the components in different systems are related definite rules of transformations.

So far we have considered only two types of physical quantities scalars and vectors. By recalling some more physical concepts, we introduce tensors. For example, in an isotropic medium, strerss $\hat{T}$ and strain $\hat{S}$ are related by the vector equation $\hat{T} = k\hat{S}$, $\hat{T}$ and $\hat{S}$ having the same direction. If the medium is not isotropic, $\hat{T}$ and $\hat{S}$ are not in general in the same direction, it is then necessary to replace the scalar k by a more general mathematical construct capable, when acting on the vector $\hat{S}$, of changing its direction as well as its magnitude. Such a mathematical construct is called 'tensor'.

Similar examples in anisotropic media can be stated such as

i) $\hat{P} = \varepsilon\,\hat{E}$ $\hat{P}$ = electric polarization

$\hat{E}$ = electric field strength

ε = electric susceptibility tensor

and

ii) $\hat{I} = \mu\hat{H}$ $\hat{I}$ = Intensity of magnetizaltion

$\hat{H}$ = Magnetic Field strength

μ = Magnetic secceptibility lenear

If we choose a general curvilinear coordniate system which could be used as in the general theory of relativity, we call the tensors as 'general tensors'. However in the case of special theory of relativity in which we deal with flat or Eucledian space, we can setup a special set of coordinate systems called 'Cartesian Coordniate systems'. These cartesian coordniate systems are related to one another through linear orthogonal transformation. The tensors in this case are called 'Cartesian tensors' which are nothing but a special class of the 'general tensors' and are concerned only with 'linear orthogonal transformation'.

9.2 Linear orthogonal transformations :

In the Euclidean space of 'n' dimensious, let us set up two cartesian coordniate systems x^i and x'^i i = l, . . . n. The coordniates of a point in the two systems are related by the following equations of transformation :

$$x'^i = \sum_{j=1}^{n} a_{ij}\ x^j \qquad \text{..... (1)}$$

which represents a linear transformation related to the rotation of axes only i.e without any change of origin. The transformation coeffecients a_{ij} are the cosines of the angles between the i th and j th axes.

Thus a transformation is said to be 'orthogonal' if it leaves the length of the displacement vector or $\sum x^{i^2}$ unaltered.

So if the linear transformation in (1) is also assumed as orthogonal, then

$$\sum_{i=1}^{n} x'^{i^2} = \sum_{k=1} x^{k^2} \qquad \text{.... (2)}$$

9.3 Tensor notation and summation convention :

The following rules of notation are used in writing tensors and tensor equations.

1) If an index appears only once in a term of the tensor equation, it is called a 'free index'.
2) If an index appears twice in any term of the tensor equation, the term stands for the sum over all possible values of that index. This is known as Einstein summation convention. The summation index is a dummy index (repeated index) and can be freely changed over to any other letter and not already present in the term. This summation index does not represent tensor character.
3) No index should appear more than two times in a term.
4) Hereafter, whenever a repeated index appears it is understood that the summation is implied over that index and hence summation symbol is dropped.

Using this notation, Eqn (1) and (2) can be written as

$$x'^i = a_{ij}\, x^j \qquad \text{.... (3)}$$

$$x^i x'^i = x^k\, x^k \qquad \text{.... (4)}$$

substituting equation 3 in equation 4,

$$x^{i'}\, x^{i'} = \left(a_{ij}\, x^j\right)\left(a_{ik}\, x^k\right) = a_{ij}\, a_{ik}\, x^j x^k \qquad \text{.... (5)}$$

equations (4) and (5) agree only if

$$\left.\begin{aligned} a_{ij}\, a_{ik} &= 1 \text{ for } j=k \\ \text{and } &= 0 \text{ for } j \neq k \end{aligned}\right\} \qquad \text{.... (6)}$$

Now introducing kronecker delta symbol as

$$\begin{aligned} \delta_{jk} &= 1 \quad j=k \\ &= 0 \quad j \neq k \end{aligned} \qquad \text{.... (7)}$$

Equation 6 can be written as

$$a_{ij}\, a_{ik} = \delta_{jk} \qquad \text{.... (8)}$$

which is the condition to be imposed to make the transformation (3) orthogonal.

To obtain the inverse transformation of (3), we proceed as follows :

$x'^i = a_{ij}\, x^j$ .... (3)

or $a_{ik}\, x'^i = a_{ij}\, a_{ik}\, x^j$

$= \delta_{jk}\, x^j = x^k$

$\therefore x^k = a_{ik}\, x'^i$ .... (9)

which is the inverse transformation to eqn (3).

The Kronecker symbol δ_{ij} in rectangular cartesian coordniates equals to $\frac{\partial x^i}{\partial x^j}$, since

$\frac{\partial x^i}{\partial x^j} = \left.\begin{matrix} 1 & i = j \\ 0 & i \neq j \end{matrix}\right\}$, Since x^i and x^j are the coordniates of the same system and hence their variations are independent of each other.

Thus $\delta_{ij} = \frac{\partial x^i}{\partial x^j}$ .... (10)

Further, it can also be viewed that δ_{ij} acts as 'index change' operator

for $\delta_{ij}\, a_j = a_i$ .... (11)

Thus the operator δ_{ij} operating on an entrity a_j has the effect of substituting i for j.

9.4 Classification of Cartesian tensors :

The tensors are classified with different orders or ranks according to the number of components they have in a space of given number of dimensions. Thus an entity having the number of its components equal to n^k in a space of n dimensions and having the requisite transformation properties will be called a tensor of rank or order k.

9.4.1 i) Zero order or Zero rank tensors

A zero order tensor will have $n^0 = 1$ component in every coordinate system. Then 'Scalars' are tensors of the zero rank.

9.4.2 ii) First rank tensors :

Any set of n quantities which take values A_i in one coordinate system and which transform in the same manner as the components of a position vector (i.e.) such that

$A'_i = a_{ij}\, A_j$ .... (12)

are said to be the components of a vector in the n-dimensional Euclidean space with respect to orthogonal cartesian coordinates.

Thus an entity having $n^1 = n$ components is a space of n dimensions is said to be a 'tensor of rank1' or equivalently an entity having the number of its components equal to the number of space, dimensions is called a 'tensor of first rank' if its components transform under linear orthogonal transformation as the components of the position vector as in equation 12.

A first rank will have $3^1 = 3$ components in 3 - dimensional space and $4^1 = 4$ components in 4 dimensional space.

A first rank tensor is called a 'Vector'.

9.4.3 iii) Second rank tensors :

A second rank tensor in an 'n' dimensional space has n^2 components.

Consider two tensors of the first rank A_i and B_i. Upon a coordinate transformation, their forms are

$$A_i' = a_{ij} A_j$$

$$B_l' = a_{lk} B_k$$

Then the n^2, quantitive $A_i B_l$ transform as

$$A_i' B_l' = a_{ij} a_{lk} A_i B_k \quad .... (13)$$

Any set of n^2 quantities which transform in the above manner (i.e) which transform as

$$T_{il}' = a_{ij} a_{lk} T_{jk} \quad .... (14)$$

are said to be the components of a 'second rank tensor'.

In 3 - dimensional space, a second order tensor has $3^2 = 9$ components and $4^2 = 16$ components in 4 - dimensional space.

A third order tensor in n-dimensional space has its law of transformation as

$$T_{ijk}' = a_{il} a_{jm} a_{kn} T_{lmn} \quad .... (15)$$

Similar law of transformations can be expressed for any order tensor.

9.5 Symmetric and Skew - Symmetric tensors :

A tensor of any rank is said to be symmetric if its two components which are obtained from each other by the interchage of any two indices are equal. Then it is said to be symmetric with respect to those two indices.

2nd order : If $A_{ij} = A_{ji}$, then A_{ij} is symmetric

3rd order : If $A_{ijk} = A_{ikj}$ then A_{ijk} is symmetric with respect to j and k indices

A tensor of any rank is said to be skew (anti) symmetric if its two components which are obtained from each other by the interchange of any two indices are equal but of oppsite sign.

2nd order : If $A_{ij} = -A_{ji}$ then A_{ij} is skew symmetric i.e., note that all the elements of the type A_{ii}. vanish in the case.

3rd order : If $A_{ijr} = -A_{jir}$, then A_{ijr} is antisymmetric with respect to i and j.

9.6 Algebraic operations on cartesian tensors :

9.6.1 i) Addition and subtraction :

The sum or difference is defined only in the case of same rank tensors and the result is once again tensor of the same rank.

$A_{ijkl} + B_{ijkl} = C_{ijkl}$

$A_{ijkl} - B_{ijkl} = D_{ijkl}$

9.6.2 ii) Multiplication :

The product of two tensors of any rank is also a tensor of rank equal to the sum of the ranks of the tensors concerned. Thus if A_{ij} and B_{klm} are two tensors of socond and third orders respectively, then their product $A_{ij} B_{klm} = C_{ijklm}$ is a tensor of rank 2+3 = 5.

9.6.3 iii) Contraction of a tensor :

This is a very typical operation with tensors of rank equal to or higher than 2.
The process of contraction of a cartenian tensor consists of putting two of its indices equal in which case the summation over that index is automatically implied.
Contraction a tensor gives another tensor which is 2 ranks lower than the original tensor.
Consider a tensor A_{ijklm} of rank 5, then its transformation is given as

$$A'_{ijklm} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, a_{mt}\, A_{pqrst} \qquad \text{.... (16)}$$

Now apply contraction with respect to l and m by putting l = m in equation 16. We obtain

$$A'_{ijkmm} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ms}\, a_{mt}\, A_{pqrst}$$

$$= a_{ip}\, a_{jq}\, a_{kr}\, a_{st}\, A_{pqrst} \qquad \text{from eqn. (8)}$$

$$= a_{ip}\, a_{jq}\, a_{kr}\, A_{pqrst}$$

which is the law of transformation of a 3rd rank tensor

i.e., $A'_{ijk} = a_{ip}\, a_{jq}\, a_{kr}\, A_{pqr}$

Note : *If an index labelled to a tensor is a repeated index, it will not contribute to the order of the tensor.*

9.7 Some fundamental properties of tensors :

i) If the components of a tensor relative to one set of co-ordinate axes are known, its components, relative to all other sets of co-ordinate axes can be known through the transformation of equations. The tensor is an entity independent of any particular system of axes.

ii) If the components of a tensor with respect to any co-ordinate system are all zero, then its components with respect to all other systems will also be zero. Further, if a tensor equation holds good in one co-ordinate system, it will hold good in every other coordinate system.

iii) Every tensor can be expressed as the sum of two parts, one symmetric and the other antisymmetric. Thus, if A_{ij} is a tensor, then

$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$ where the first bracket is symmetric and the second bracket is an antisymmetric tensor.

9.8 Special carterian tensors :

Tensors which have the same components in all the frames of reference are called as 'invaniant' or 'isotropic' tensors. We consider two such tensors, namely, Kronecker tensor and Alternate or Epsilon tensor.

9.8.1 (i) Kronecker tensor :

In the orthogonal carterian coordinate system, we know that the three basis unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ have the property

$$\hat{e}_i.\hat{e}_j = \left.\begin{matrix} 1 & i=j \\ 0 & i\neq j \end{matrix}\right\}$$

or $= \delta_{ij}$ (Kroecker delta) .... (17)

It can be shown that δ_{ij} is a tensor of rank 2. If it is a tensor of rank 2, it must transform as follows :

$$\delta'_{ij} = a_{il}\, a_{jk}\, \delta_{lk} = a_{il}\, a_{jl}$$

$= \delta_{ij}$ from equation 8 .... (18)

Hance δ_{ij} is a tensor of rank 2..

9.8.2. (ii) The alternate or Epsilon tensor :

We know the relation among the three orthonormal basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as

$\hat{e}_i \times \hat{e}_i = \hat{0}$ and $\hat{e}_i \times \hat{e}_j = \hat{e}_k$ .... (19)

where i, j, k take values from 1 to 3.

So the direction of the vector $\hat{e}_i \times \hat{e}_j$ is along $\hat{e}_k$ and its margnitude is given by $\hat{e}_i \times \hat{e}_j . \hat{e}_k$. Or it can be seen that

$\hat{e}_i \times \hat{e}_j . \hat{e}_k$ = 0 if any two of the indices i, j, k are equal.
= +1 if the indices i, j, k are unequal and in cyclic order
= -1 if the indices i, j, k are unequal and not in cyclic order. ...... (20)

Introducing a symbol ϵ_{ijk} known as alternate or 'epsilon' tensor, it is defined as follows :

ϵ_{ijk} = 0 if any of the two indices are equal.
= +1 if the inidces are unequal and in cyclic order
= -1 if the indices are unequal and not in cyclic order .... (21).

This is a third order anti - symmetric tensor which transforms as

$$\epsilon'_{ijk} = a_{il}\, a_{jm}\, a_{kn}\, \epsilon_{lmn}$$ .... (22).

If has $3^3 = 27$ components in 3 - dimentional space of which six only are non zero components.

The ϵ - tensor can be used to write the cross product of two vectors $\hat{A}$ and $\hat{B}$

Let $\hat{D} = \hat{A} \times \hat{E}$, then

$D_1 = A_2 B_3 - A_3 B_2 = \in_{123} A_2 B_3 + \in_{132} A_3 B_2 \quad \because \in_{123} = 1 \quad \in_{132} = -1$

$= \in_{1jk} A_j B_k$ (using summation convention)

Similarly $D_2 = \in_{2jk} A_j B_k$ and $D_3 = \in_{3jk} A_j B_k$

or $D_i = \in_{ijk} A_j B_k$ ..... (23)

This is the representation of the cross product of two vectors using $\in$ - tensor notation.

In the same way, the scalar triple product of $\hat{A}, \hat{B}, \hat{C}$ can be written as

$r = \hat{C} . (\hat{A} \times \hat{B})$

$= \in_{ijk} C_i A_j B_k$ in view of equations (20) and (21). ...... (24)

9.9. Examples of tensors in elasticity :

9.9.1 (i) Strain tensor :

When we apply some forces to a solid elastic body, the particles of that body undergo relative displacements. That is, the configuration of the body changes or the body is in a 'strained state'. The ragid body translation and rotations do not produce any strain.

For a mathematical description of the 'Strain' produced in the body, consider a rectangular. carterian coordinate system (x^1, x^2, x^3). When the forces are applied to the body, each point of the body is subjected to a dispalcement vector $\hat{u}$ which is a function of the position vector

$\hat{r} = (x^1, x^2, x^3)$ (i.e.) $u_i = u_i (x^1, x^2, x^3)$

$= u_i (x^i)$ ..... (25)

Consider two adjacent points P $(\hat{r})$ and Q $(\hat{r} + d\hat{r})$. Let $\hat{u}(\hat{r})$ be the dispalcement suffered by P and $\hat{u}(\hat{r} + d\hat{r})$ correspondingly by Q. Let the new positions be P' and Q' as shown in figure 1.

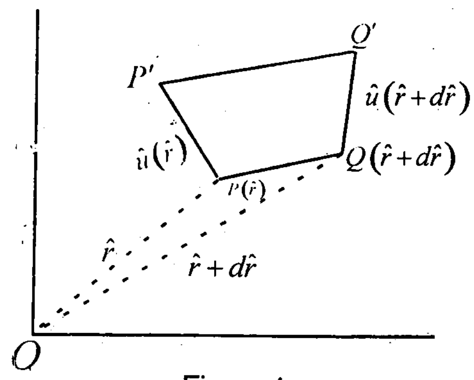


Figure 1.

Then $\hat{u}\,(\hat{r}+d\hat{r})$ of the point Q may be written as

$$\hat{u}\,(\hat{r}+d\hat{r}) = u_i\,(x^i+dx^i) \qquad i = 1,2,3$$

$$\approx u_i\,(x^i) + \frac{\partial u_i}{\partial x^j}\,dx^j$$

or $u_i\,(x^i+dx^i) - u_i(x^i) \approx \frac{\partial u_i}{\partial x^j}\,dx^j$ ..... (26)

$\frac{\partial u_i}{\partial x^j}$ is a rank tensor being the gradient of the components of the vector $\hat{u}$. so eqn. - (26) can be written as

$$\frac{\partial u_i}{\partial x^j}\,dx^j = dx^j\left[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}\right)}_{\text{Symmetric}} + \underbrace{\left(\frac{\partial u_i}{\partial x^j}-\frac{\partial u_j}{\partial x^i}\right)}_{\text{AntiSymmetric}}\right] \qquad \text{..... (27)}$$

$$= dx^j\,(e_{ij}+\phi_{ij}) \qquad \text{..... (28)}$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}\right) = e_{ji}$ ..... (29)

and $\phi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j}-\frac{\partial u_j}{\partial x^i}\right) = -\,\phi_{ji}$ ..... (30)

With the antisymmetric tensor ϕ_{ij}, a vector ψ_k can be associated as

$$\phi_{ij} = \epsilon_{ijk}\,\psi_k \qquad \text{...... (31)}$$

where $\psi_k = \frac{1}{2}\,\epsilon_{ijk}\,\phi_{ij}$ ...... (32)

So from (28), $\phi_{ij}\,dx^j = \epsilon_{ijk}\,\psi_k\,dx^j$ ..... (33)

The RHS of 33 can be considered as the i th component of the vector product $\hat{\psi} \times d\hat{r}$ (vide eqn. (23) which represents the displacement due to a rotation $\hat{\psi}$. Hence the part $\phi_{ij}\,dx^j$ of the displacement of the point Q due to the 'rigid body rotation' will not contribute to the 'strain'. So in the RHS of equation (28), $e_{ij}\,.\,dx^j$ is called 'pure strain' while the quantities e_{ij} constitute the components of the 'strain tensor' given by

$$e_{ij} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x^1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x^2} + \dfrac{\partial u_2}{\partial x^1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x^3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x^1} + \dfrac{\partial u_1}{\partial x^2}\right) & \dfrac{\partial u^2}{\partial x^2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x^3} + \dfrac{\partial u_3}{\partial x^2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x^1} + \dfrac{\partial u_1}{\partial x^3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x^2} + \dfrac{\partial u_2}{\partial x^3}\right) & \dfrac{\partial u_3}{\partial x^3} \end{pmatrix} \quad \ldots\ldots (34)$$

It is easily seen that the 'Strain tensor' e_{ij} is a second rank symmetric tensor and has only six independent components.

Physical significance of the components of the 'strain tensor' :

Consider a volume element $dx^1 dx^2 dx^3$ in the shape of a rectangular parallelopiped ABCDEFG as shown in the figure 2.

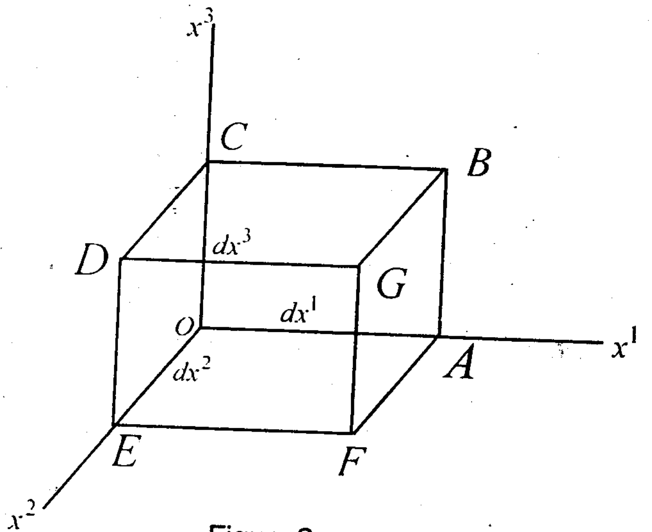


Figure 2.

If each point of the body is subjected to a dispalcement $\hat{u}$ varying from point to point, $d\hat{u}\,(du_1, du_2, du_3)$ is the corresponding dispalcement in the volume element. The change in the x^1 - direction alone is

$$du_1 = \frac{\partial u_1}{\partial x^1}\, dx^1$$

or change per unit length = $\dfrac{\partial u_1}{\partial x^1}$

Thus $\dfrac{\partial u_1}{\partial x^1}$ is the linear strain in the x^1 - direction. If then follows that $\dfrac{\partial u_2}{\partial x^2}$ and $\dfrac{\partial u_3}{\partial x^3}$ are the inear strains in the x^2 and x^3 directions respectively. So the physical significance of the diagonal term in the 'strain' tensor (34) is that they represent 'elongations' or 'linear strains' along the three rectangular axes x^1, x^2, x^3.

Next, we understand the significance of the off-diagonal elements of the strain tensor.

Consider the plane OABC alone of the parallelopiped and let it be deformed into a parallelogram $OAB'C'$ by applying a tangential force parallel to CB and keeping OA fixed as shown in figure 3.

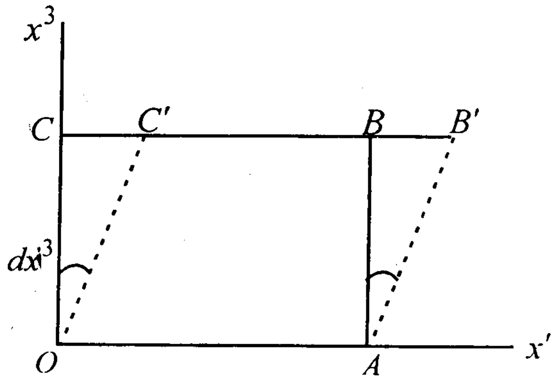


Figure 3,

The point C moves to C' and B to B' and all other points between AB and OC slide over from rectangle to the parallelogram. This type of deformation is termed as 'shear' and is measured by tan $\angle COC'$ equal to CC'/OC

$$\text{(i.e) shearing strain} = \frac{CC^1}{OC} = \frac{\partial u_1}{\partial x^3} = \frac{\text{displacement in } x' \text{direction}}{\text{Change in the } x^3 \text{ coordinate}} \quad \text{..... (35)}$$

In this shearing strain, there is no deformation along the x^2 - direction. The only other shearing strain in which there is again no deformation along x^2 - direction is

$$\frac{AA'}{OA} = \frac{\partial u_3}{\partial x'} = \frac{\text{displacement in } x^3 \text{ direction}}{\text{Change in the } x' \text{ coordinate}} \text{ as shown in the figure 4.} \quad \text{...... (36)}$$

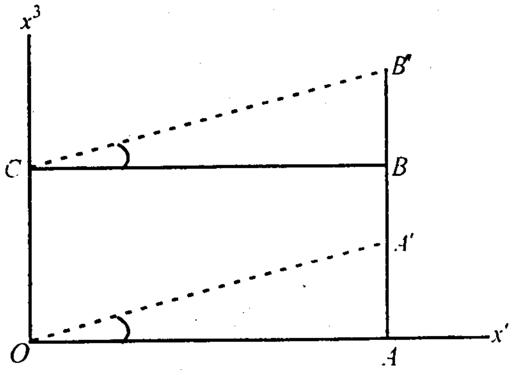


Figure 4.

But since shear is a scalar, the total shearing strain in $x^1 x^3$ - plane is $\frac{\partial u_1}{\partial x^3}+\frac{\partial u_3}{\partial x^1}$ Similarly $\frac{\partial u_2}{\partial x^3}+\frac{\partial u_3}{\partial x^2}$ and $\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}$ are the total shearing strain components in $x^2 x^3$ and $x^1 x^2$ - planes respectively which are all twice the off-diagonal terms in the strain tensor.

9.9.2 (ii) Stress tensor :

The stress is defined as the internal force per unit area acting on a deformed body. The stress may be directed normally or tangentially to the surfaces on which they act. If the deforming forces act normally to a given area of an elastic medium, they produce pure elongations and in that case, the stresses are called "tensile or normal stresses" and if the deforming forces act tangentially, 'shearing stresses' are produced.

In figure 2, consider the face OCDE of the parallelopiped in the x^2x^3 - plane. Let a component F_1, of the total force $\hat{F}$ act normal to the face area $A_1 = dx^2 dx^3$. Then the stress (σ_1) acting normal to this face is defined by

$$\sigma_{11} = \frac{\partial F_1}{\partial A_1}$$

In a similar manner, stresses normal to faces OABC and OEFA are defined respectively as

$$\sigma_{22} = \frac{\partial F_2}{\partial A_2} \quad \text{and} \quad \sigma_{33} = \frac{\partial F_3}{\partial A_3}$$

Thus the only 'normal stresses' acting on the faces of the volume element are $\sigma_{11}, \sigma_{22}, \sigma_{33}$.

Next consider tangential forces acting on the face OCDE, i.e., the forces acting along x^2 and x^3 directions. Then the shearing stress in x^2 - direction acting on a plane perpendicular to the x^1 - direction is defined by $\sigma_{12} = \frac{\partial F_2}{\partial A_1}$ ...... (37)

When the force is along x^3 - direction acting on the same face OCDE, the shearing stress σ_{13} is given by $\sigma_{13} = \frac{\partial F_3}{\partial A_1}$ ...... (38)

(__Note :__ OCDE plane is $x^2 x^3$ - plane. There are two axial directions namely x^2 - axis and x^3 - axis. But there is a unique normal i.e., x^1 - axis. Thus there are two shearing stress components σ_{12} and σ_{13} as defined above)

In a similar manner, for the other two planes also, two more pairs of shearing stress components can be defined as $\sigma_{21} = \frac{\partial F_1}{\partial A_2}, \sigma_{23} = \frac{\partial F_3}{\partial A_2}$

and $\sigma_{32} = \frac{\partial F_2}{\partial A_3}, \sigma_{31} = \frac{\partial F_1}{\partial A_3}$

So the total stress can be represented by a matrix as

$$T = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \text{...... (39)}$$

It can be shown that shearing stresses on mutually perpendicular planes are always equal i.e., $\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$, Once again consider the parallelopiped (fig. 2) with sides dx^1, dx^2 and dx^3. Now for the volume element to be in static equilibrium, the angular accelaration vanishes or the total torque must be zero. Or the balance of moments of forces requires that $(\sigma_{12} dx^2 dx^3) dx^1 = (\sigma_{21} dx^1 dx^3) dx^2$ or $\sigma_{12} = \sigma_{21}$. Similarly $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$ - (40)

Thus the stress matmix is symmetrical and needs only six independent components for its complete specification.

Next we prove that the matrix

$$T = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial A_1} & \frac{\partial F_2}{\partial A_1} & \frac{\partial F_3}{\partial A_1} \\ \frac{\partial F_1}{\partial A_2} & \frac{\partial F_2}{\partial A_2} & \frac{\partial F_3}{\partial A_2} \\ \frac{\partial F_1}{\partial A_3} & \frac{\partial F_2}{\partial A_3} & \frac{\partial F_3}{\partial A_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial A_1} \\ \frac{\partial}{\partial A_2} \\ \frac{\partial}{\partial A_3} \end{pmatrix} (F_1 \quad F_2 \quad F_3) \qquad \text{...... (41)}$$

is a tensor.

It is known that area is a vector quantity, the direction being considered along the normal. So A_1, A_2, A_3 which are already defined as areas must obey the transformation law of a vector.

i.e., $A_i' = a_{ij} A_j$ ..... (42)

and $\frac{\partial}{\partial A_i'} = \left(\frac{\partial}{\partial A_j}\right)\left(\frac{\partial A_j}{\partial A_i'}\right)$ or $\frac{\partial}{\partial A_i'} = a_{ij} \frac{\partial}{\partial A_j}$ ...... (43)

which is the vector transformation law. (i.e.) $\begin{pmatrix} \frac{\partial}{\partial A_1} \\ \frac{\partial}{\partial A_2} \\ \frac{\partial}{\partial A_3} \end{pmatrix}$ is a vector

It is obvious that the force (F_1, F_2, F_3) is a vector. So T in (41) should transform like a tensor of order 2 since (41) is an open product of two vectors.

Thus the stress matrix (41) is a second order symmetric tensor with diagonal terms representing 'normal' stresses and the off - diagnal terms as 'shearing or tangential' stresses.

9.10. Summary of the Lesson

Basic definition of scalars and vectors are given. By considering some physical examples in anisotropic media, the concept of a tensor has been introduced. Distinguishing between 'general tensors' and 'cartesian tensors' this lesson is devoted mainly to 'cartesian tensors' which are concerned with linear orthogonal transformations.

By briefly explaining the linear orthogonal transformation, cartesian tensor notation and summation convention have been introduced and explained in detail.

Classifying the cartesian tensors, transformation law as a definition of a tensor is given. Symmetry properties and algebra of tensors have been explaned with exampels. The process of contraction of a tensor in clearly given along with some propertion of of tensors.

Special cartenian tensors such as kroncker tensor and epsilon tensor are expalnied along with their properties.

Two specific examples of 'Strain tensor' and 'Stress tensor' in elasticity are considered giving their formation and the physical significances.

9.11 Key terminology

Scalars - Vectors - Tensors - Carterian tensors - Linear orthogonal transfermation - Free index - Repeated index - Index change operator - rank of a tensor - Contraction of a tensor - Symetric tensor - Kronecker tensor - Epsilon tensor - Strain tensor - Stress tensor - Shear and normal stress (Strain).

9.12 Self - Assessment Questions

1. Distinguish between 'general' and cartesian tensors. Explain the summation convention used in tensors.

2. If the law of transformation of a cartesian tensor is $\overline{A}_i = a_{ij} A_j$, work out the inverse transformations.

3. Define symmetric and antisymmetric tensors and show that every second order tensor can be expressed as a sum of symmetric and antisymmetric tensors.

4. Explain multiplication of two tensors and contraction of a cartesian tensor. Contract the tensor A_{ijklm} twice and show that the result is a vector.

5. What are invariant tensors and explain Kronecker and explain tensors.

6. Explain $\in$-tensor and how do you represent the cross product of two vectors using $\in$ tensor.

7. Define Kronecker and epsilon tensors and prove the following :

   (i) $\delta_{ij} \in_{ijl} = 0$ (ii) $\in_{ijl} \in_{kjl} = 2\delta_{ik}$

   (iii) $\left(\delta_{im}\,\delta_{kp} + \delta_{ip}\,\delta_{km}\right) w_{ik} = w_{mp} + w_{pm}$

8. Define stress and bring out the stress tensor of a defined body. Explain the physical significance.

9. The displacement of points in an elastically strained cubic crystal sample $1 \times 1 \times 1\ cm^3$ in size are

$$u_1 = \left(4x_1 + 3x_2 - 5x_3\right)10^{-4}\ cm$$

$$u_2 = \left(7x_1 - 13x_2 + 4x_3\right)10^{-4}\ cm$$

$$u_3 = \left(9x_1 - 2x_2 + 4x_3\right)10^{-4}\ cm$$

Find the small displacement tensor, strain tensor and rotation tensor.

9.13 Reference Books :

1. P.P. Gupta, RPS Yadav and GS Malik 'Mathematical Physics' Kedarnath Ramnath, Meerut, 1980.

2. F.A. Hinchey, 'Vectors and tensors for Engineers and Scientists', Wiley Eastern Ltd., Delhi, 1976.

3. H. Margenau and G.M. Murphy, 'The Mathematics of Physica and Chemistry', Affiliated East-West Press Pvt. Ltd., New Delhi, 1956.

4. C.E. Weatherburn, 'Riemannian Geometry and the tensor Calculus', Cambridge University Press, 1957.

Unit - III

Lesson - 10

General Tensors

Objective of the Lesson :

> To define general tensors in a curvilinear space.

> To give laws of transformations of several kinds of tensors.

> To develop algebra of tensors and their properties.

> To define line element and fundamental tensor in Riemannian space.

> To study the nature and usages of the fundamental tensors.

Structure of the lesson :

10.1 Introduction :

It is seen in the last lesson about the cartesian tensors which are characterized by linear orthogonal transformations with constant coefficients i.e., those among cartesian co-ordinate system in Euclidean space. In this chapter we propose to generalise the cartesian tensors applicable to more general type of spaces namely, curved spaces of which Euclidean space is only a particular space.

The use of more general type of co-ordinates called oblique (non orthogonal) curvilinear co-ordinates will lead to the general tensors, which are used in the formulation of the general theory of relativity.

10.2 Transformation of co-ordinates :

Consider a set of n single valued functions $\phi^i(x^1, x^2, \ldots\ldots x^n), i = 1, 2, \ldots\ldots n$ in an n - dimensional space V_n. If the functional determinant is not equal to zero, the system of n equations

$$\bar{x}^i = \phi^i(x^1, x^2, \ldots\ldots\ldots x^n) \qquad \text{--------} (1)$$

can be solved for the x's in terms of the $\bar{x}$'s giving

$$x^i = \psi^i(\bar{x}^1, \bar{x}^2, \ldots\ldots\ldots \bar{x}^n) \qquad \text{--------} (2)$$

The equations (1) and (2) define a transformation of co-ordinates and they enable us to pass from either system of co-ordinates to the other.

Since the functions $\bar{x}^i$ are independent it is to be noted that

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \bar{\delta}^i_j = 1 \quad \text{for} \quad i = j$$

$$= 0 \quad \text{for} \quad i \neq j \qquad \text{----------} (3)$$

called as Kronecker delta symbol.

10.3 Contravariant Vector :

Firstly, we consider the infinitesimal displacement vector as an example of the contravariant vector.

Let P be a point of V_n whose co-ordinates are x^i in the one system and $\bar{x}^i$ in the other; and let Q be an adjacent point whose co-ordinates are $x^i + dx^i$ in the former system and $\bar{x}^i + d\bar{x}^i$ in the latter. The two sets of differentials are, of course connected by the equations

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \qquad \text{------- (4)}$$

Thus the infinitesimal displacement PQ, whose components in the co-ordinate system x^i are the differentials dx^i is an example of a contravariant vector. Its componets in the co-ordinate system $\bar{x}^i$ are the differentials $d\bar{x}^i$ and the componets in these two systems are connected by the set of equations (4).

Generalizing this, the definition of contravariant vector is given as follows. If two sets of functions u^i and $\bar{u}^i$ (i=1,2,......n) are connected by the relations

$$\bar{u}^i = u^j \frac{\partial \bar{x}^i}{\partial x^j} \qquad \text{------ (5)} \quad (i=1,2,.........n)$$

the quantities u^i are said to be the components of a contravariant vector in the co-ordinate system x^i, while $\bar{u}^i$ are the components of the same vector in the system $\bar{x}^i$.

10.4 Scalar invariants :

The term scalar invariant or scalar denotes any function which is not changed by transformation of co-ordinates. If such a function is represented by $A(x^1, x^2, ... x^n)$ in the system $\bar{x}^i$ and by $\bar{A}(\bar{x}^1, \bar{x}^2, ... \bar{x}^n)$ in the system $\bar{x}^i$ then $\bar{A}(\bar{x}^1, \bar{x}^2, ... \bar{x}^n) = A(x^1, x^2, ... x^n)$

The partial derivatives of this invariant with respect to the co-ordinates in the system x^i are the n functions $A_i = \dfrac{\partial A}{\partial x^i}$ ------- (6)

Its partial derivatives in the system $\bar{x}^i$ are given by $\bar{A}_i = \dfrac{\partial \bar{A}}{\partial \bar{x}^i}$.

10.5 Covariant Vector :

We consider the gradient of scalar invariant function as an example of covariant vector as follows. If $A(x^1, x^2, ... x^n)$ is an invariant function and correspondingly $\bar{A}(\bar{x}^1, \bar{x}^2, ... \bar{x}^n)$ in the x^i & $\bar{x}^i$ systems respectively, then $$\overline{A}_i = \frac{\partial \overline{A}}{\partial \overline{x}^i} = \frac{\partial A}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^i}$$

(or) $$\overline{A}_i = A_k \frac{\partial x^k}{\partial \overline{x}^i} \qquad \text{------- (7)}$$

The vector, whose components in the x's are the partial derivatives A_i, is called the gradient of the scalar A, and is denoted briefly by grad A. The gradient, thus defined, is an example of a covariant vector.

More generally, if V_i are n functions of the x's and $\overline{V}_i$ n functions of the $\overline{x}$'s connected by the relations

$$\overline{V}_i = V_k \frac{\partial x^k}{\partial \overline{x}^i} \qquad \text{------ (8)}$$

we say that the V_i are the components of a covariant vector in the system x^i, and $\overline{V}_i$ are the components of the same vector in the $\overline{x}^i$.

Note : *It should be observed that the index of a covariant vector is written as a superscript, and that of a covariant vector as a subscript.*

Theorem :

If the sum $u^i v_i$ is an invariant, and the quantities u^i are the components of an arbitary contravariant vector, then the quantities v_i are the components of a covariant vector. Or if v_i are the components of an arbitary covariant vector, u^i are the components of a contravariant vector.

Proof :

Since the given sum is invariant we have

$$\overline{u}^i \overline{v}_i - u^j v_j = 0 \qquad \text{------ (9)}$$

As the quantities u^i are the components of the arbitrary contravariant vector, equation (9) can be written as

$$u^j \frac{\partial \bar{x}^i}{\partial x^j} \bar{v}_i - u^j v^j = 0$$

or $$u^j \left(v_i \frac{\partial \bar{x}^i}{\partial x^j} - v_j \right) = 0 \qquad \text{------ (10)}$$

If this holds for an arbitrary contravariant vector, the coefficients of the quantities u^j must all be zero showing that v_j are the components of a covariant vector.

Similarly, for an arbitrary covariant vector, the theorem can be proved.

10.6 Tensors of second order :

Let $\bar{A}^{ij}$ $(i, j = 1,2,\ldots\ldots\ldots n)$ be a set of n^2 functions of the variables x^i. Also let $\bar{A}^{ij}$ be n^2 functions of the $\bar{x}^i$'s, connected with the former by equations of the form

$$\bar{A}^{ij} = A^{kl} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} \qquad \text{-------- (11)} \qquad (i,j,l = 1,2,\ldots\ldots\ldots n)$$

Then the quantities A^{ij} are said to be the components of a contravariant tensor of second order in the coordinate system x^i, and $\bar{A}^{ij}$ are the components of the same tensor in the system $\bar{x}^i$.

A covariant tensor of the second order is such that its n^2 components A_{ij} in the system x^i are connected with its components $\bar{A}^{ij}$ in any other system $\bar{x}^i$ by equations of the form

$$\bar{A}_{ij} = A_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \qquad \text{------- (12)}$$

A mixed tensor of the second order has both covariant and contravariant characteristics, its components A^i_j in the x's being connected with its components $\bar{A}^i_j$ in the $\bar{x}^s$ by the relations

$$\bar{A}^i_j = A^k_l \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} \qquad \text{-------- (13)}$$

10.7 Tensors of any order :

Tensors of higher than the second order are similarly defined. A contravariant tensor of order k obeys the following law of transformation as

$$\overline{A}^{pq\ldots\ldots i} = A^{ab\ldots\ldots d}\frac{\partial \overline{x}^p}{\partial x^a}\frac{\partial \overline{x}^q}{\partial x^b}\ldots\ldots\ldots\ldots\frac{\partial \overline{x}^i}{\partial x^d} \quad \text{------ (14)}$$

(the no. of indicies for each component being k)

Similarly a covariant tensor of order m, whose components in any other system $\overline{x}^i$ are given by

$$\overline{A}_{hi\ldots\ldots l} = A_{\alpha\beta\ldots\ldots\delta}\frac{\partial x^\alpha}{\partial \overline{x}^h}\frac{\partial x^\beta}{\partial \overline{x}^i}\ldots\ldots\ldots\ldots\frac{\partial x^\delta}{\partial \overline{x}^l} \quad \text{------ (15)}$$

(the no. of indices for each component being m)

Lastly, the n^{m+k} quantities $A^{pq\ldots\ldots t}_{hi\ldots\ldots l}$ k being the number of superscripts and m the number of subscripts, are the x-components of a mixed tensor of order k+m, provided its components in any order system $\overline{x}^i$ are given by

$$\overline{A}^{pq\ldots\ldots t}_{hi\ldots\ldots l} = A^{ab\ldots d}_{\alpha\beta\ldots\partial}\frac{\partial \overline{x}^p}{\partial x^a}\ldots\ldots\ldots\frac{\partial \overline{x}^t}{\partial x^d}\frac{\partial x^\alpha}{\partial \overline{x}^h}\ldots\ldots\ldots\frac{\partial x^\delta}{\partial \overline{x}^l} \quad \text{------ (16)}$$

(for each component, the no. of contravariant indices being k and the no. of covariant indices being m.)

Note : *A contravariant (or covariant) vector is a contravariant (or covariant) tensor of the first order. The number of different indices denotes the order or rank of the tensor.*

10.8 Symmetric and skew-symmetric tensors :

The covariant tensor of the second order, whose components are A_{ij}, is said to be symmetric if $A_{ij} = A_{ji}$ for all values of i and j; and similarly for a contravariant tensor of the second order. A symmetric tensor of the second order has the number of different components corresponding to different indices (off diagonal elements) as $\frac{n^2-n}{2}$ and the number corresponding to a repeated index (diagonal elements) is n thus leading to the total number of different components as $\frac{n^2-n}{2}+n$ or $\frac{n(n+1)}{2}$. A tensor of order higher than the second is said to the symmetric with respect to any two indices when two components, obtainable from each other by interchanging these indices are equal.

A tensor is said to be anti or skew-symmetric with respect to two covariant or two contravariant indices, when the two components obtained from each other by interchanging the indices differ only in sign. Thus, a second order covariant tensor whose components are

A_{ij} is skew-symmetric if $A_{ij} = -A_{ji}$ for all values of i and j. Consequently, $A_{ii} = 0$. Therefore, except as regards sign, there are only $\frac{n(n-1)}{2}$ different non-zero components.

10.9 Addition of two tensors :

The sum (or the difference) of two tensors of the same kind is a tensor of that kind.

Q : Show that any tensor of the second order may be expressed as the sum of a symmetric tensor and a skew-symmetric tensor.

Sol : The components A_{ij} of the given tensor can always be written as

$$A_{ij} = \underbrace{\frac{1}{2}(A_{ij} + A_{ji})}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A_{ij} - A_{ji})}_{\text{skew-symmetric}}$$

Since the quantities $A_{ij} + A_{ji}$ are the components of a symmetric tensor (i.e., $A_{ij} + A_{ji} = A_{ji} + A_{ij}$) and similarly $A_{ij} - A_{ji}$ are the components of a skew-symmetric tensor (i.e., $A_{ij} - A_{ji} = -\overline{A_{ij} - A_{ji}}$),

the result follows.

10.10 Outer or open product of two tensors :

The product of two tensors is a tensor whose order is the sum of the orders of the two tensors. This tensor is called the open or outer product of the two tensors.

Q : Determine the nature and order of the open product of the two tensors A^{ij} and B_k .

Sol : The open product of the two given tensors A^{ij} and B_k is given by

$$(A^{ij} B_k) = \left(\frac{\partial x^i}{\partial \overline{x}^\alpha}\frac{\partial x^j}{\partial \overline{x}^\beta}\overline{A}^{\alpha\beta}\right)\left(\frac{\partial \overline{x}^\gamma}{\partial x^k}\overline{B}_\gamma\right) \quad \text{(Using the respective transformations)}$$

$$= \frac{\partial x^i}{\partial \overline{x}^\alpha}\frac{\partial x^j}{\partial \overline{x}^\beta}\frac{\partial \overline{x}^\gamma}{\partial x^k}(\overline{A}^{\alpha\beta}\overline{B}_\gamma)$$

or $C^{ij}_k = \frac{\partial x^i}{\partial \overline{x}^\alpha}\frac{\partial x^j}{\partial \overline{x}^\beta}\frac{\partial \overline{x}^\gamma}{\partial x^k}\overline{C}^{\alpha\beta}_\gamma$ where $C^{ij}_k = A^{ij} B_k$ and $\overline{C}^{\alpha\beta}_\gamma = \overline{A}^{\alpha\beta}\,\overline{B}_\gamma$

This represents the law of transformation of a mixed tensor of order 3 (2 contravariant and 1 covariant indices).

10.11 Contraction of a tensor :

Any mixed tensor may be contracted giving a tensor whose order is less by 2 than that of the original tensor. The process of 'Contraction' consists in putting one of the covariant indices equal to one of the contravariant and performing the summation indicated.

Q: In how many ways can tensor T^{ab}_{rst} be contracted? By taking one way of contraction, determine the nature and order of the resulting tensor.

Sol : The given 5th order mixed tensor T^{ab}_{rst} can be contracted in 6 ways and in each way by putting a=r or s or t or b=r or s or t. Let us consider one way of contraction by putting a=r.

The law of transformation for the mixed tensor T^{ab}_{rst} is

$$T^{ab}_{rst} = \frac{\partial x^a}{\partial \bar{x}^\alpha} \frac{\partial x^b}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial \bar{x}^n}{\partial x^t} \bar{T}^{\alpha\beta}_{lmn}$$

Applying contraction process by putting a=r, we get

$$T^{rb}_{rst} = \frac{\partial x^r}{\partial \bar{x}^\alpha} \frac{\partial x^b}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial \bar{x}^n}{\partial x^t} \bar{T}^{\alpha\beta}_{lmn}$$

$$= \frac{\partial x^b}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial \bar{x}^n}{\partial x^t} \bar{\delta}^l_\alpha \bar{T}^{\alpha\beta}_{lmn}$$

$$= \frac{\partial x^b}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial \bar{x}^n}{\partial x^t} \bar{T}^{\alpha\beta}_{\alpha mn}$$

$$\because \frac{\partial x^r}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^l}{\partial x^r} = \frac{\partial \bar{x}^l}{\partial \bar{x}^\alpha} = \bar{\delta}^l_\alpha$$

which shows the law of transformation of a mixed tensor of order 3 (one contravarient and two covariant indices). T^{rb}_{rst} has 'r' as the dummy (repeated) index and it will not contribute to the order of the mixed tensor. Thus it represents a 3rd order mixed tensor (3 distinct indices).

10.12 Compounding or inner product of two tensors :

Two tensors may be 'Compounded' by first forming their outer product and then contracting it with respect to an index of the one with an index of opposite character of the other. The result is called the inner product of two tensors. Its order is less by 2 than the sum of the orders of the original tensors.

Thus the tensors A^i_j and B^k_{lm} may be compounded in the above manner in 3 different ways giving the tensors.

$A^i_j B^j_{lm}$, $A^i_j B^k_{im}$, $A^i_j B^k_{li}$

which are 3rd order mixed tensors.

Note : *Putting k = l or m or i=j is forbidden as per the definition of the inner product.*

10.13 Quotient Law :

If $A^{ab...c}_{ij...k}$ are functions of the x's and $\overline{A}^{ab...c}_{ij...k}$ function of the $\overline{x}$'s such that $u^i \overline{A}^{ab...c}_{ij...k}$ and $\overline{u}^i \overline{A}^{ab...c}_{ij...k}$ are components of a tensor in the coordinate systems x^i and $\overline{x}^i$ respectively and when u^i and $\overline{u}^i$ are components of an arbitray contravariant vector in these systems, then the given functions are components of a tensor of the type as indicated by the indices.

Proof : From the given data, if follows that $\overline{u}^i \overline{A}^{ab...c}_{ij...k} = u^l A^{pq...r}_{lm...n} \frac{\partial \overline{x}^a}{\partial x^p} \cdots \frac{\partial \overline{x}^c}{\partial x^r} \frac{\partial x^m}{\partial \overline{x}^j} \cdots \frac{\partial x^n}{\partial \overline{x}^k}$ noting that i is a dummy index.

In the RHS expression, we may put $u^l = \frac{\partial x^l}{\partial \overline{x}^i} \overline{u}^i$ as it is given as arbitray contravaniant vector.

Then, since the equations are true for an arbitrary contravaniant vector, the coefficients of $\overline{u}^i$ in the two members are equal. Thus we have equations expressing that the A's are components of a tensor of the type as indicated by the indices.

Note : *The theorem in which equations 9 and 10 are involved is considered to be the particular case of the quotient law.*

10.14 Fundamental tensor :

The distance ds between adjacent points whose rectangular cartesian co-ordinates are (x, y, z) and (x + dx, y + dy, z + dz) is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

More generally, for any system of oblique curvilinear co-ordinates u, v, w, we have

$$ds^2 = adu^2 + bdv^2 + cdw^2 + 2f\,dv\,dw + 2g\,dw\,du + 2h\,du\,dv$$

where a, b, c, f, g, h, are functions of the co-ordinates. Extending this concept to space of n dimensions, the infinimitesimal distance ds between the adjacent points,

x^i and $x^i + dx^i$ (i = 1, 2 . .. n) is given by the relation

$$ds^2 = g_{ij}\, dx^i dx^j \quad (i, j = 1, 2 \ldots n) \qquad \text{----------- (17)}$$

where the coefficients g_{ij} are functions of the co-ordinates x^i. The quadratic differential form in the R.H.S. of (17) is called a Riemannian metric or line element. If $g_{ij} \neq 0$ then the space in said to be a Riemannian space. In particular, if g_{ij} 's are all independent of x^i, the space is Euclidian.

Since the diffrentials dx^i are components of a contravariant vector and the ds^2 is from its nature a scalar invariant, the functions g_{ij} must be the components of a covariant tensor of second order. This tensor is called the fundamental covariant tensor and its reciprocal tensor g^{ij} is called the fundamental contravarient tensor. If the value of $|g_{ij}|$ is denoted by g, then the determinant $|g^{ij}|$ is equal to 1/g. Further $g_{ij}\, g^{jk} = \delta_i^k$

Q : In the Riemannian metric, show that the functions g_{ij} are the components of a symmetric covariant tensor of order two.

Soln. : We know tha the Riemannian metric given by

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1,2 \ldots n)$$

is a scalar invariant and hence it follows that

$$g_{ij}\, dx^i\, dx^j = \bar{g}_{ab}\, d\bar{x}^a\, d\bar{x}^b = \bar{g}_{ab} \frac{\partial \bar{x}^a}{\partial x^i} \cdot \frac{\partial \bar{x}^b}{\partial x^j} \cdot dx^i\, dx^j \qquad \text{----------- (18)}$$

As the differentials are the components of contravariant vector, so by virtue of the quotient law, we have $g_{ij} = \dfrac{\partial \bar{x}^a}{\partial x^i} \cdot \dfrac{\partial \bar{x}^b}{\partial x^j}\, \bar{g}_{ab}$ ----------- (19)

which shows that g_{ij} is a covariant tensor of order 2. From equation (17) it is obvious that

$$ds^2 = g_{ij}\, dx^i\, dx^j = g_{ji}\, dx^j\, dx^i \qquad \text{----------- (20)}$$

which shows that g_{ij} is a symmetric tensor.

10.15 Magnitude of a vector :

Equation (17) can be written as

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1 \qquad \text{----------- (21)}$$

which can be understood that $\frac{dx^i}{ds}$ are the components of a unit contravariant vector and equation 21 as the scalar product of the unit vector with itself.

By analogy with equation 21, we define the length or magnitude u of any contravariant vector, whose components are u^i by equation $u^2 = g_{ij}\, u^i\, u^j$ -------- (22)

Similarly, the magnitude v of a covariant vector, whose components are v_i is defined by $v^2 = g^{ij}\, v_i\, v_j$ ---------- (23)

10.16 Associate covariant and contravariant vectors :

The vector $g_{ij}\, u^j$ is a covariant vector which is said to be associate to u^i by means of the fundamental tensor. It is usually denoted by u_i. Thus $u_i = g_{ij}\, u^j$ --------- (24)

Similarly, the vector v^i defined by

$$v^i = g^{ij}\, v_j \qquad \text{--------- (25)}$$

is the contravaiant vector associate to v_i by means of the fundamental tensor

Thus the magnitude u of any contravariant vector as given in equation 22 can be written as

$$u^2 = g_{ij}\, u^i\, u^j = (g_{ij}\, u^i)\, u^j$$

$= u_j\, u^j$ by virtue of equation (24). --------- (26)

= Scalar (Compounding a corviant vector with its contravaiant part).

It is convenient to refer to u^i and u_i as the contravariant and covaraint components respectively of one and the same vector $\hat{u}$. Equation (26), can also be written in the vector notation as $\hat{u}.\hat{u} = u^2 = u_i u^i = g_{ij} u^i u^j = g^{ij} u_i u_j$

Thus we can bring, the connection between vector and tensor notations by defining the scalar product of two vectors $\hat{u}$ and $\hat{v}$ as

$$\hat{u}.\hat{v} = g_{ij} u^i v^j = u_j v^j = u^i\, v_i = uv \cos\theta \qquad \text{-------- (27)}$$

where θ is the angle between the two vectors $\hat{u}$ and $\hat{v}$.

Or $\cos\theta = \dfrac{g_{ij}u^i v^j}{vu} = \dfrac{g_{ij}u^i v^j}{\sqrt{g_{ij}u^i u^j}\sqrt{g_{ij}v^i v^j}}$ -------- (28)

The condition of orthogonality of the two vectors $\hat{u}$ and $\hat{v}$ is $\hat{u} \cdot \hat{v} = 0$. which is equivalent to $g_{ij}u^i v^j = 0$. ------------ (29)

It $\hat{u}$ is a unit vector, then $\hat{u} \cdot \hat{u} = 1$ or

$g_{ij}u^i u^j = 1$ ------------ (30)

Note : *1) The process of obtaining the associate vector by compounding with one of the fundamental tensors is referred to as lowering the superscript, or 'raising' the subscript.*

Ex : $g_{ij}u^j = u_i$ *(lowering the superscript)*

$g^{ij}u_i = u^j$ *(raising the subscript)*

2) By compounding any tensor with the fundamental tensor, the base remains unaltered while the position of the index is changed :

Ex : In $g_{ij}u^j = u_i$, *the result has the same base u and the superscript has changed to subscript. But, if* A_{ij} *is not a fundamental tensor,* $A_{ij}u^j = B_i$ *but can not be* u_i.

10.17 Problems :

1. If θ is the inclination of the vectors $\hat{u}$ and $\hat{v}$ show that

$$\sin^2\theta = \frac{(g_{hi}\,g_{jk} - g_{hk}\,g_{ij}).(u^h u^i v^j v^k)}{g_{hi}\,g_{jk}\,u^h u^i\,v^j v^k)}$$

Soln : Now the RHS of the problem

$$= \frac{(g_{hi}\,u^h\,u^i)\,(g_{jk}\,v^j\,v^k) - (g_{hk}\,u^h\,v^k)\,(g_{ij}\,u^i v^j)}{(g_{hi}\,u^h\,u^i)\,(g_{jk}\,v^j\,v^k)}$$

$$= \frac{u^2 v^2 - (u\,v\cos\theta)\,(uv\cos\theta)}{u^2\,v^2}$$ since $g_{hi}u^h u^i = u^2$ and $g_{hk}u^h v^k = uv\cos\theta$

$$= \frac{u^2 v^2 -(1-\cos^2\theta)}{u^2 v^2} = \sin^2\theta$$

Hence the result.

2. Show that for a rectangular system of coordinates the raising and lowering of suffix leaves the components unaltered in three dimensional space.

Soln : We have in the rectangular, coordinate system (Cartesian coordinate system), the line element (square of the infinitesimal vector) as given by

$$ds^2 = dx^{1^2} + dx^{2^2} + dx^{3^2} \text{ in 3-dimensional Space.} \quad \text{----- (31)}$$

where $x^1 = x$, $x^2 = y$; $x^3 = z$.

Comparing (31) with the general line element (21).

We have $g_{11} = g_{22} = g_{33} = 1$ and $g_{12} = g_{13} = g_{23} = 0$

$$\therefore g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

We thus have

$$g^{11} = \frac{\text{Co factor of } g_{11} \text{ in } g}{g} = \frac{1}{1} = 1$$

Similarly $g^{22} = g^{33} = 1$

Which follows that $g_{ij} = g^{ij}$

Thus the raising or lowering of suffix leaves the components of a tensor unaltered in the rectangular coordinate system.

Note :- *There is no difference between contravariant tensor or covariant tensor in the cartesian coordinate systems as there is no distinction between fundamental contravariant or covariant tensors.*

3. Show that the symmetry properties of a tensor are invariant.

Soln :- Consider a symmetry property of a tensor as $A_{ijk} = A_{jik}$. Then we have

to show that $\overline{A}_{ijk} = \overline{A}_{jik}$.

As per definition, it follows that

$$\overline{A}_{ijk} = \frac{\partial x^{\alpha}}{\partial \overline{x}^{i}} \frac{\partial x^{\beta}}{\partial \overline{x}^{j}} \frac{\partial x^{\gamma}}{\partial \overline{x}^{k}} A_{\alpha\beta\gamma} \quad \text{-------(1)}$$

and $$\overline{A}_{jik} = \frac{\partial x^{\alpha}}{\partial \overline{x}^{j}} \frac{\partial x^{\beta}}{\partial \overline{x}^{i}} \frac{\partial x^{\gamma}}{\partial \overline{x}^{k}} A_{\alpha\beta\gamma}$$

or also $$= \frac{\partial x^{\beta}}{\partial \overline{x}^{j}} \frac{\partial x^{\alpha}}{\partial \overline{x}^{i}} \frac{\partial x^{\gamma}}{\partial \overline{x}^{k}} A_{\alpha\beta\gamma} \quad \text{-------(2)}$$

as α, β, γ are dummy or repeated indices.

As we have considered $A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$, from (1) and (2), it is obvious that $\overline{A}_{ijk} = \overline{A}_{jik}$ which proves the invariant property of the symmetry property.

4. If A_{ij} is a skew-symmetric tensor, show that $(B^i_p B^j_q + B^i_q B^j_p) A_{ij} = 0$.

Soln : From the given problem,

$$(B^i_p B^j_q + B^i_q B^j_p) A_{ij} = (B^j_p B^i_q + B^j_q B^i_p) A_{ji} \quad \because \text{i and j are repeated indices}$$

$$= -(B^i_p B^j_q + B^i_q B^j_p) A_{ij} \quad \because A_{ij} \text{ is skew-symmetric}$$

or $$2(B^i_p B^j_q + B^i_q B^j_p) A_{ij} = 0$$

Hence the result.

5. Show that

(i) $A_{\alpha\beta} B^{\alpha\beta} = A^{\alpha\beta} B_{\alpha\beta}$

(ii) $A_{ij} B^{kj} = A^j_i B^k_j$

Soln : (i) We know that

$$A_{\alpha\beta} = g_{\alpha i}\, g_{\beta j}\, A^{ij}$$

and $B^{\alpha\beta} = g^{\alpha i} g^{\beta j} B_{ij}$

$$\therefore A_{\alpha\beta} B^{\alpha\beta} = g_{\alpha i} g^{\alpha i} g_{\beta j} g^{\beta j} A^{ij} B_{ij}$$

$$= (g_{\alpha i} g^{\alpha i} A^{ij})(g_{\beta j} g^{\beta j} B_{ij})$$

$$= (g^{\alpha i} A^{j}_{\alpha})(g_{\beta j} B^{\beta}_{i})$$

$$= A^{ij} B_{ij} = A^{\alpha\beta} B_{\alpha\beta}$$

(ii) We know that $A_{ij} = g_{\beta j} A^{\beta}_{i}$

and $B^{kj} = g^{j\alpha} B^{k}_{\alpha}$

$$\therefore A_{ij} B^{kj} = g_{\beta j} g^{j\alpha} A^{\beta}_{i} B^{k}_{\alpha}$$
$$= \delta^{\alpha}_{\beta} A^{\beta}_{i} B^{k}_{\alpha} = A^{\alpha}_{i} B^{k}_{\alpha}$$

6. Consider, in E_2, the tensor A_{ij} whose covariant components relative to a cartesian coordinate system (x,y) are $A_{11} = x^2$, $A_{12} = A_{21} = xy$; $A_{22} = y^2$

Calculate the components in the coordinate system (r,θ) which is defined as

$$x = r\cos\theta,\ y = r\sin\theta.$$

Soln :

$$\text{Let} \left.\begin{matrix} x^1 = x & ;\ x^2 = y \\ \bar{x}^1 = r & ;\ \bar{x}^2 = \theta \end{matrix}\right\} \text{ ---------- (1)}$$

From the transformation between the coordinate system, (i.e.,) $x = r\cos\theta$, $y = r\sin\theta$

we have
$$\begin{pmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \text{ ------(2)}$$

The law of transformation of a covariant tensor is

$$\overline{A}_{ij} = \frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j} A_{ab}$$
$$= \frac{\partial x^1}{\partial \overline{x}^i}\frac{\partial x^1}{\partial \overline{x}^j} A_{11} + \frac{\partial x^2}{\partial \overline{x}^i}\frac{\partial x^2}{\partial \overline{x}^j} A_{22} + \frac{\partial x^1}{\partial \overline{x}^i}\frac{\partial x^2}{\partial \overline{x}^j} A_{12} + \frac{\partial x^2}{\partial \overline{x}^i}\frac{\partial x^1}{\partial \overline{x}^j} A_{21} \quad \text{-----(3)}$$

$$\therefore \overline{A}_{11} = \frac{\partial x^1}{\partial \overline{x}^1}\frac{\partial x^1}{\partial \overline{x}^1} A_{11} + \frac{\partial x^2}{\partial \overline{x}^1}\frac{\partial x^2}{\partial \overline{x}^1} A_{22} + \frac{\partial x^1}{\partial \overline{x}^1}\frac{\partial x^2}{\partial \overline{x}^1} A_{12} + \frac{\partial x^2}{\partial \overline{x}^1}\frac{\partial x^1}{\partial \overline{x}^1} A_{21}$$
$$= \text{Cos}\theta.\text{Cos}\theta.(r\text{Cos}\theta)^2 + (\text{Sin}\theta)(\text{Sin}\theta)(r\text{Sin}\theta)^2 + 2\text{Cos}\theta.\text{Sin}\theta(r\text{Cos}\theta)(r\text{Sin}\theta) \quad \text{from (2)}$$
$$= r^2 (\text{Cos}^2\theta + \text{Sin}^2\theta)^2 = r^2$$

Since $A_{12} = A_{21} = xy = r^2 \text{Sin}\theta\text{Cos}\theta$,

$$\therefore \overline{A}_{12} = \frac{\partial x^1}{\partial \overline{x}^1}\frac{\partial x^1}{\partial \overline{x}^2} A_{11} + \frac{\partial x^2}{\partial \overline{x}^1}\frac{\partial x^2}{\partial \overline{x}^2} A_{22} + \frac{\partial x^1}{\partial \overline{x}^1}\frac{\partial x^2}{\partial \overline{x}^2} A_{12} + \frac{\partial x^2}{\partial \overline{x}^1}\frac{\partial x^1}{\partial \overline{x}^2} A_{21}$$
$$= -r\,\text{Sin}\theta.\text{Cos}\theta.(r\text{Cos}\theta)^2 + (r\,\text{Sin}\theta\text{Cos}\theta)(r\text{Sin}\theta)^2 + r\text{Cos}^2\theta.r^2\,\text{Sin}\theta\text{Cos}\theta - r\text{Sin}^2\theta.r^2\,\text{Sin}\theta\text{Cos}\theta$$
$$= 0 = \overline{A}_{21}$$

Lastly,

$$\overline{A}_{22} = (-r\,\text{Sin}\theta)^2 (r\text{Cos}\theta)^2 + (r\,\text{Cos}\theta)^2\,(r\text{Sin}\theta)^2 + r^2\text{Sin}\theta\text{Cos}\theta.r^2\,\text{Sin}\theta\text{Cos}\theta - r^2\text{Sin}\theta\text{Cos}\theta.r^2\text{Sin}\theta\text{Cos}\theta$$
$$= 0$$

So, $(\overline{A}_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & 0 \end{pmatrix}$

7. Transform the line element $ds^2 = dx^2 + dy^2 + dz^2$ into spherical polar coordinates and find the volume element.

Soln : The transformation equations from the cartesian to polar coordinates are

$$x = r\text{Sin}\theta\text{Cos}\varphi, \quad y = r\text{Sin}\theta\text{Sin}\varphi, \quad z = r\text{Cos}\theta \quad \text{----(1)}$$

If a point (x^1, x^2, x^3) in cartesian system becomes $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ in polar system,

then $x^1 = x, \quad x^2 = y, \quad x^3 = z$

$\overline{x}^1 = r, \quad \overline{x}^2 = \theta, \quad \overline{x}^3 = \phi$

From the given line element, the fundamental tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ------(2)$$

The law of transformation of g_{ij} is

$$\bar{g}_{ab} = \frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b} g_{ij}$$

$$\therefore \bar{g}_{11} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^1} g_{ij}$$

$$= \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^1} g_{11} + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^1} g_{22} + \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^1} g_{33}$$

in view of the vanshing elements in g_{ij} tensor (2)

$$= \left(\frac{\partial x}{\partial r}\right)^2.1 + \left(\frac{\partial y}{\partial r}\right)^2.1 + \left(\frac{\partial z}{\partial r}\right)^2.1$$

$$= (\text{Sin}\,\theta\,\text{Cos}\,\varphi)^2 + (\text{Sin}\,\theta\,\text{Sin}\,\varphi)^2 + (\text{Cos}\,\theta)^2 \quad \text{from (1)}$$

$$= 1$$

Similarly

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2.1 + \left(\frac{\partial y}{\partial \theta}\right)^2.1 + \left(\frac{\partial z}{\partial \theta}\right)^2.1$$

$$= (r\,\text{Cos}\,\theta\,\text{Cos}\,\phi)^2.1 + (r\,\text{Cos}\,\theta\,\text{Sin}\,\phi)^2.1 + (-r\,\text{Sin}\,\theta)^2.1$$

$$= r^2$$

and $$\bar{g}_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2\,\text{Sin}^2\theta$$

Further, $$\bar{g}_{12} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^2} g_{ij}$$

$$= \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2}.1 + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2}.1 + \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}.1$$

in view of (2)

$$= (\sin\theta\cos\phi).(r\cos\theta\cos\phi) + (\sin\theta\sin\phi).(r\cos\theta\sin\phi) + \cos\theta\,(-r\sin\theta)$$

$$= 0$$

Similarly, $\bar{g}_{13} = 0 = \bar{g}_{23}$

$$\therefore\ \overline{ds}^2 = 1.(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta.(d\phi)^2 \quad ----- (3)$$

is the required line element in polar coordinates.

Now $(dx)^2+(dy)^2+(dz)^2 = ds^2 = (\overline{ds})^2 = (1.dr)^2+(r\,d\theta)^2+(r\sin\theta\,d\phi)^2$

$\therefore$ The volume element dx dy dz can be expressed in spherical polar coordinates as

$$dx\,dy\,dz = (1.dr)\,(r\,d\theta)\,(r\sin\theta\,d\phi)$$
$$= r^2\sin\theta\,dr\,d\theta\,d\phi$$

8. Show that the velocity and acceleration at any point are contravariant vectors.

Solution :- Let $x^i(t)$ be the coordinate as a function of time of a moving particle. Then the velocity component is

$$V^i = \frac{dx^i}{dt} \quad ------ (1)$$

If the coordinate system from x^i is changed to $\bar{x}^i$, then

$$V^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial \bar{x}^j}\frac{d\bar{x}^j}{dt} = \frac{\partial x^i}{\partial \bar{x}^j}\bar{V}^j \quad ----- (2)$$

which shows that the velocity components obey the law of transformation of a contravariant vector.

Taking the time derivative of (2), it follows that

$$\frac{dV^i}{dt} = \frac{\partial x^i}{\partial \bar{x}^j}\frac{d\bar{V}^j}{dt} \quad -------- (3)$$

Here, a subtle difference in understanding eq - (2) is necessary. The coordinates x^i in $\frac{dx^i}{dt}$ are the coordinates of a particle in motion, while the coefficients $\frac{\partial x^i}{\partial \bar{x}^j}$ only denote a relation between two fixed coordinate systems, which is independent of time.

Thus eq - (3) gives a law of transformation of a contravariant vector $\frac{dV^i}{dt}$ (i.e.,) the acceleration.

9. Find the components of a vector in polar coordinates whose contravariant componenets in cartesian system are $\ddot{x}$ and $\ddot{y}$ in a V_2 space.

Soln : Let $x^1 = x, \quad x^2 = y$

$$\bar{x}^1 = r, \quad \bar{x}^2 = \theta \quad \text{-------(1)}$$

and $A^1 = \ddot{x}, \quad A^2 = \ddot{y}$

We have the transformation

$$\left.\begin{array}{ll} x = r\cos\theta, & y = r\sin\theta \\ \text{or} \quad r^2 = x^2 + y^2; & \theta = \tan^{-1}\frac{y}{x} \end{array}\right\} \quad \text{------(2)}$$

From the law of transformation of a contravariant vector, we have

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \qquad (i, j = 1,2) \quad \text{-------(3)}$$

Now $\bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^j} A^j = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2$

$$= \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y}$$

$$= \frac{x\ddot{x}}{r} + \frac{y\ddot{y}}{r} \qquad \text{from(2)}$$

$$= \frac{x\ddot{x} + y\ddot{y}}{r} \quad \text{--------- (4)}$$

and $\bar{A}^2 = \frac{\partial \bar{x}^2}{\partial x^1} A^1 + \frac{\partial \bar{x}^2}{\partial x^2} A^2$

$$= \frac{\partial \theta}{\partial x}\ddot{x} + \frac{\partial \theta}{\partial y}\ddot{y}$$

$$= -\frac{y}{r^2}\ddot{x} + \frac{x}{r^2}\ddot{y} \qquad \text{from}(2)$$

$$= \frac{x\ddot{y} - y\ddot{x}}{r^2} \qquad \text{---------} \ (5)$$

Now from (2)

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}.\frac{dr}{dt} + \frac{\partial x}{\partial \theta}.\frac{d\theta}{dt} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad \text{----} \ (6)$$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \qquad \text{----} \ (7)$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \qquad \text{--------} \ (8)$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \qquad \text{----} \ (9)$$

Making use of (7) and (9) along with (2),

$\frac{x\ddot{x} + y\ddot{y}}{r}$ gives $\ddot{r} - r\dot{\theta}^2$

and $\frac{x\ddot{y} - y\ddot{x}}{r^2}$ gives $\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}$

Hence the required components in polar coordinates are

$$\overline{A}^1 = \ddot{r} - r\dot{\theta}^2$$

$$\overline{A}^2 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}$$

10. Consider a coordinate system (u, v, w) which is related to the cartesian coordinates by $x = vw$, $y = uw$, $z = uv$.

Obtain the metric in terms of u, v, w.

Solution :- We know the line element and the corresponding metric tensor as

$$\left.\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ \text{and} \quad g_{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}\right\} \qquad \text{--------} \ (1)$$

We take

$$\left.\begin{array}{lll} x^1 = x, & x^2 = y, & x^3 = z \\ \bar{x}^1 = u, & \bar{x}^2 = v, & \bar{x}^3 = w \\ x = vw, & y = uw, & z = uv \end{array}\right\} \qquad \text{---------} \quad (2)$$

We know the law of transformation of the fundamental covariant tensor as

$$\bar{g}_{ij} = \frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial x^b}{\partial \bar{x}^j} g_{ab} \qquad \text{--------} \quad (3)$$

In view of the metric tensor in (1), we can write

$$\begin{aligned} \bar{g}_{11} &= \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^1}.1 + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^1}.1 + \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^1}.1 \\ &= 0 + w^2 + v^2 \qquad \text{from (2)} \end{aligned}$$

By cyclic symmetry, we can have

$$\begin{aligned} \bar{g}_{22} &= u^2 + w^2 \\ \bar{g}_{33} &= v^2 + u^2 \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{g}_{12} &= \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2}.1 + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2}.1 + \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}.1 \\ &= 0 + 0 + vu \end{aligned}$$

Also $\bar{g}_{23} = wv$ and $\bar{g}_{31} = uw$ by cyclic symmetry.

Therefore the line element in (u v w) system is given by

$$ds^2 = \bar{g}_{ij}.d\bar{x}^i d\bar{x}^j \qquad (i,j=1,2,3)$$

Where $$\bar{g}_{ij} = \begin{pmatrix} v^2 + w^2 & uv & uw \\ uv & u^2 + w^2 & vw \\ uw & vw & u^2 + v^2 \end{pmatrix}$$

and $|\bar{g}_{ij}| = 4u^2 v^2 w^2$.

10.18 Summary of the lesson :

In the general curvilinear coordinate system, definitions of contravariant, covariant and mixed tensors of various orders have been defined by giving the laws of transformations and examples. Products of tensors and contraction of a tensor are defined.

The line element and the fundamental tensor are defined. The importance of the fundamental tensor for raising or lowering the indices is stressed. Several examples are worked for better understanding.

Some simple rules regarding the indices of tensors are noteworthy.

(i) If an index appears only once in any term, it has a definite value - any value between 1 and n if n-dimensional space is considered. Such an index is called a free index. This index should match in all terms throughout the equation. This means that if a free index occurs as a contravariant index in one term, it should occur as a contravariant index in each term of the equation.

(ii) An index which is repeated and over which summation is implied is called a dummy index. It can be replaced by any other index which does not appear in the same term. This index may occur only in some of the terms of an equation. When it occurs in a term, it should occur twice, once in a contravariant position and once in a covariant position.

(iii) No index should occur more than twice in any term.

(iv) When a coordinate differential such as ∂x^i occurs in a term, i is to be regarded as a contravariant index if ∂x^i occurs in the numerator and as a covariant index if it occurs in the denominator. Thus, as an example, in $\frac{\partial x^i}{\partial \bar{x}^a}$, i is to be treated as contravariant while a general and contravariant is to be treated as covariant index.

10.19 Key terminology

General tensors, Contravariant tensors, Scalar Invariant, covariant and mixed tensors, free index, dummy index, open product, inner product, contraction, Riemannian space, fundamental tensor, Associate Contravariant and covariant vectors.

10.20 Self Assessment questions

(1) In a Euclidean manifold E_2, the covariant components in an orthogonal cartesian system are

$$A_{11} = x^2, A_{12} = A_{21} = 0, A_{22} = y^2$$

Obtain the components of the same vector in the parabolic coordinate system (u, v) given by $x = \frac{u-v}{2}$; $y = \sqrt{uv}$.

(2) A function A (p,q,r,s) of coordinates x^i transforms to another system of coordinates as

$$\bar{A}(abcd) = \frac{\partial x^p}{\partial \bar{x}^a} \frac{\partial \bar{x}^b}{\partial x^q} \frac{\partial \bar{x}^c}{\partial x^r} \frac{\partial \bar{x}^d}{\partial x^s} A(p,q,r,s)$$

Is it a tensor ? If so, give its nature and rank.

(3) If B_{ij} is an arbitary covariant tensor and $A(p,i)\, B_{ij} = C_{pj}$ where C_{pj} is a tensor, then prove that $A(p,i)$ is a mixed tensor.

(4) State the law of transformation of a mixed tensor. Show, by example, how the transformation is affected when the tensor is subjected to a contraction.

(5) If A_{ij} is an anti symmetric tensor and S_{ij} is a symmetric tensor, find whether are not any of the following tensors is anti symmetric or symmetric ?

(i) $A_{ij}\, A_{ik}$ (ii) $A_{ij}\, S_{ik}$ (iii) $S_{ij}\, S_{ik}$ (iv) $A_{ij}\, S_{ik} - S_{ij}\, A_{ik}$

(6) Transform $ds^2 = dx^2 + dy^2 + dz^2$ into cylindrical coordinates.

(7) In V_2- space, the components of a contravariant vector are $\dot{x}$, $\dot{y}$. Transform them into polar coordinates.

(8) Show that $A_{\alpha\beta}$ is a tensor if its inner product with an arbitrary mixed tensor C^{β}_{γ} is a tensor.

10.21 Reference Books :

1. C.E. Weatherburn 'An introduction to Riemannian geometry and the tensor calculus'
(Cambridge University Press, 1957)

2. F.A. Hinchey 'Vectors and tensors for engineers and scientists'
(Wiley Eastern Ltd., New Delhi, 1976)

3. A.W. Joshi 'Matrices and tensors in Physics'
(Wiley Eastern Ltd., New Delhi, 1975)

4. B.S. Rajput 'Mathematical Physics'
(Pragathi Prakashan, Meerut, 1999)

Unit III
Lesson – 11

CHRISTOFFEL SYMBOLS

Objectives of the Lesson:

- To introduce the concept of symbols
- To explain the notation and properties of the symbols.
- To obtain the law of transformation of the symbols.
- To give precautionary notes on the calculation of the symbols.
- To work out some important examples.

Structure of the Lesson:

11.1 Introduction:

As the fundamental tensor is in general a function of the coordinates, its partial derivatives and their combinations often appear in the scientific problems.

The derivatives of a tensor with respect to the coordinate system are not in general tensors. In order to construct differential calculus for tensors, it is necessary to define an operation of differentiation in such a way that derivative of a tensor is another tensor. This task is most easily accomplished in terms of certain combinations of the partial derivatives of the metric tensor. Grouping of such derivatives are denoted by symbols called Christoffel symbols.

11.2 Christoffel symbols:

Special symbols *[k, ij] and $^{*}\begin{Bmatrix} k \\ ij \end{Bmatrix}$ called the Christoffel symbols of the first and second kinds

(* These can also be denoted as [ij, k] and Γ^k_{ij} respectively)

are given by $$[k, ij] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad \text{----------} \quad (1)$$

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kh}[h, ij] \quad \text{----------} \quad (2)$$

The expression on the right of (2) being summed with respect to the repeated index h. The three indices in Christoffel symbol of first kind are considered to be covariant type indices. In the symbol of the second kind, the upper index is to be regarded as contravariant type index and the lower one as covariant type index.

From their definition, it follows that both functions are symmetric in the indices i and j and not with any other pair of indices.

From equation (1), it can also be seen that

$$\frac{\partial g_{ik}}{\partial x^j} = [k, ij] + [i, jk] \quad \text{----------} \quad (3)$$

Multiplying equation (2) with g_{lk} on both sides,

we have

$$g_{lk}\begin{Bmatrix} k \\ ij \end{Bmatrix} = g_{lk}\, g^{kh}\,[h, ij] = \delta^h_l\,[h, ij] = [l, ij] \quad \text{----------} \quad (4)$$

11.3 Derivative of g^{ij}:

Q: Express the partial derivative of the fundamental contravariant tensor in terms of the Christoffel symbols of the second kind.

OR

Show that $$\frac{\partial g^{ik}}{\partial x^j} = -\,g^{hk}\begin{Bmatrix} i \\ hj \end{Bmatrix} - g^{hi}\begin{Bmatrix} k \\ hj \end{Bmatrix}$$

Solution: we know the identity

$$g^{hi}\, g_{hl} = \delta^i_l \quad \text{----------} \quad (5)$$

Differentiating equation (5) with respect to x^j, we have

$$g^{ih}\frac{\partial g_{hl}}{\partial x^j} + g_{hl}\frac{\partial g^{ih}}{\partial x^j} = 0 \qquad \because \delta^i_l = \begin{Bmatrix} 1 & i = l \\ 0 & i \neq l \end{Bmatrix}.$$

Multiplying by g^{lk}, and summing with respect to l,

we have

$$\delta_h^k \frac{\partial g^{ih}}{\partial x^j} = - g^{lk} g^{ih} \frac{\partial g_{hl}}{\partial x^j}$$

or $$\frac{\partial g^{ik}}{\partial x^j} = - g^{lk} g^{ih} ([h, lj] + [l, hj]) \quad \text{from equation (3)}$$

$$= - g^{lk} \begin{Bmatrix} i \\ lj \end{Bmatrix} - g^{ih} \begin{Bmatrix} k \\ hj \end{Bmatrix} \quad \text{from equation (2)}$$

Replacing the dummy index l by h we may write the relation

$$\frac{\partial g^{ik}}{\partial x^j} = - g^{hk} \begin{Bmatrix} i \\ hj \end{Bmatrix} - g^{hi} \begin{Bmatrix} k \\ hj \end{Bmatrix} \quad \text{----------} \ (6)$$

Note: Having come to this extent, it is very important to consolidate our ideas about the fundamental tensor.

In the previous lesson, while dealing with associate vectors, it has been mentioned that

$$g_{ij} u^j = u_i$$

which means that $g_{ij} u^j$ is a covariant vector associate to u^i by means of the fundamental tensor. Or, it can be understood that g_{ij} is used to lower the index j in u^j to i in u_i . Or it can also be viewed that the inner product of the fundamental tensor with a tensor results in the tensor with the same base and indices balanced.

Example:

$g_{ij} u^j = u_i$ base: u

$g_{ij} A^j_{kl} = A_{ikl}$ base: A

$g^{ij} B^k_{ilm} = B^{jk}_{lm}$ base: B

$g^{kh} [h, ij] = \begin{Bmatrix} k \\ ij \end{Bmatrix}$ base: Christoffel symbol

$g_{lk} \begin{Bmatrix} l \\ ij \end{Bmatrix} = [k, ij]$ base: Christoffel symbol.

But $g^{hk} \begin{Bmatrix} i \\ hj \end{Bmatrix}$ cannot be further simplified giving $\begin{Bmatrix} ki \\ j \end{Bmatrix}$ because there is no such Christoffel symbol defined. Further, inner products with other than fundamental tensor cannot exhibit such property that the result also will have the same base.

Example: $A_{ij}\, u^j \neq u_i$ but it is $A_{ij}\, u^j = B_i$

Where B is a base different from u.

Q: Prove that $\dfrac{\partial}{\partial x^j} \log \sqrt{g} = \begin{Bmatrix} i \\ ij \end{Bmatrix}$

Solution: Since gg^{ik} is the cofactor of g_{ik} in the determinant g, from the rule of differentiating a determinant, we have

$$\frac{\partial g}{\partial x^j} = gg^{ik}\,\frac{\partial g_{ik}}{\partial x^j} = gg^{ik}\,[k, ij] + [i, kj]$$

$$= g\begin{Bmatrix} i \\ ij \end{Bmatrix} + g\begin{Bmatrix} k \\ kj \end{Bmatrix} = 2g\begin{Bmatrix} i \\ ij \end{Bmatrix} \qquad \text{where } g = |\,g_{ij}\,|$$

or $$\frac{\partial}{\partial x^j} \log \sqrt{g} = \begin{Bmatrix} i \\ ij \end{Bmatrix} \qquad \text{----------} \quad (7)$$

11.4 Laws of transformation of the 3 – index symbols:

Q: Express the second derivatives of the x's with respect to the $\overline{x}$'s in terms of their first derivatives and the Christoffel symbols for both systems of coordinates.

OR

Obtain the law of transformation of the Christoffel symbols

OR

Show that the Christoffel symbols do not denote components of a tensor.

Solution: On differentiating the law of transformation

$$\overline{g}_{ij} = g_{ab}\,\frac{\partial x^a}{\partial \overline{x}^i}\,\frac{\partial x^b}{\partial \overline{x}^j} \qquad \text{----------} \quad (8)$$

with respect to $\overline{x}^k$, we have

$$\frac{\partial \overline{g}_{ij}}{\partial \overline{x}^k} = \frac{\partial g_{ab}}{\partial x^c}\,\frac{\partial x^a}{\partial \overline{x}^i}\,\frac{\partial x^b}{\partial \overline{x}^j}\,\frac{\partial x^c}{\partial \overline{x}^k} + g_{ab}\,\frac{\partial^2 x^a}{\partial \overline{x}^i \partial \overline{x}^k}\,\frac{\partial x^b}{\partial \overline{x}^j} + g_{ab}\,\frac{\partial x^a}{\partial \overline{x}^i}\,\frac{\partial^2 x^b}{\partial \overline{x}^j \partial \overline{x}^k} \qquad \text{----------} \quad (9)$$

By cyclic permutation of the indices i, j, k, we further have

$$\frac{\partial \overline{g}_{jk}}{\partial \overline{x}^i} = \frac{\partial g_{bc}}{\partial x^a}\,\frac{\partial x^b}{\partial \overline{x}^j}\,\frac{\partial x^c}{\partial \overline{x}^k}\,\frac{\partial x^a}{\partial \overline{x}^i} + g_{bc}\,\frac{\partial^2 x^b}{\partial \overline{x}^j \partial \overline{x}^i}\,\frac{\partial x^c}{\partial \overline{x}^k} + g_{bc}\,\frac{\partial x^b}{\partial \overline{x}^j}\,\frac{\partial^2 x^c}{\partial \overline{x}^k \partial \overline{x}^i} \qquad \text{----------} \quad (10)$$

and

$$\frac{\partial \overline{g}_{ki}}{\partial \overline{x}^j} = \frac{\partial g_{ca}}{\partial x^b}\,\frac{\partial x^c}{\partial \overline{x}^k}\,\frac{\partial x^a}{\partial \overline{x}^i}\,\frac{\partial x^b}{\partial \overline{x}^j} + g_{ca}\,\frac{\partial^2 x^c}{\partial \overline{x}^k \partial \overline{x}^j}\,\frac{\partial x^a}{\partial \overline{x}^i} + g_{ca}\,\frac{\partial x^c}{\partial \overline{x}^k}\,\frac{\partial^2 x^a}{\partial \overline{x}^i \partial \overline{x}^j} \qquad \text{----------} \quad (11)$$

since a, b, c are dummy indices and the fundamental tensor is symmetric the combination of the equations (i.e) $\frac{1}{2}[10 + 11 - 9]$ gives

$$\frac{1}{2}\left(\frac{\partial \overline{g}_{jk}}{\partial \overline{x}^i} + \frac{\partial \overline{g}_{ki}}{\partial \overline{x}^j} - \frac{\partial \overline{g}_{ij}}{\partial \overline{x}^k}\right) = \frac{1}{2}\left(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}\right)\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j}\frac{\partial x^c}{\partial \overline{x}^k} + g_{ab}\frac{\partial x^a}{\partial \overline{x}^k}\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j} \quad \text{-(12)}$$

with the understanding that the last term of (9) and the second term of (11), and similarly the second term of (10) and the last term of (11) get cancelled.

By the definition of Christoffel symbols, (12) takes the form

$$\left[\overline{k, ij}\right] = [c, ab]\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j}\frac{\partial x^c}{\partial \overline{x}^k} + g_{ab}\frac{\partial x^a}{\partial \overline{x}^k}\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j} \quad \text{---------- (13)}$$

This is the law of transformation of Christoffel symbols of the first kind connecting the functions in $\overline{x}^i$ and x^i systems. But for the presence of the last term, (13) would have represented the law of transformation of a third rank covariant tensor.

On multiplying the two sides of the equation (13) by the corresponding sides of the identity

$$\overline{g}^{lk}\frac{\partial x^d}{\partial \overline{x}^l} = g^{dh}\frac{\partial \overline{x}^k}{\partial x^h} \quad \text{---------- (14)}$$

and summing with respect to the repeated indices, we obtain

$$\left[\overline{k, ij}\right]\overline{g}^{lk}\frac{\partial x^d}{\partial \overline{x}^l} = [c, ab]\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j}\frac{\partial x^c}{\partial \overline{x}^k}\frac{\partial \overline{x}^k}{\partial x^h}g^{dh} + g_{ab}\,g^{dh}\frac{\partial \overline{x}^k}{\partial x^h}\frac{\partial x^a}{\partial \overline{x}^k}\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j} \quad \text{---------- (15)}$$

$$\overline{\begin{Bmatrix} l \\ ij \end{Bmatrix}}\frac{\partial x^d}{\partial \overline{x}^l} = [c, ab]\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j}\delta^c_h\,g^{dh} + g_{ab}\,g^{dh}\,\delta^a_h\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j}$$

$$= g^{dc}[c, ab]\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j} + g_{ab}\,g^{da}\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j}$$

$$= \begin{Bmatrix} d \\ ab \end{Bmatrix}\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j} + \delta^d_b\frac{\partial^2 x^b}{\partial \overline{x}^i \partial \overline{x}^j}$$

$$= \begin{Bmatrix} d \\ ab \end{Bmatrix}\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j} + \frac{\partial^2 x^d}{\partial \overline{x}^i \partial \overline{x}^j} \quad \text{----------- (16)}$$

Multiplying this equation by $\frac{\partial \overline{x}^h}{\partial x^d}$ and summing with respect to d, we get

$$\overline{\begin{Bmatrix} h \\ ij \end{Bmatrix}} = \begin{Bmatrix} d \\ ab \end{Bmatrix}\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^b}{\partial \overline{x}^j}\frac{\partial \overline{x}^h}{\partial x^d} + \frac{\partial^2 x^d}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial \overline{x}^h}{\partial x^d} \quad \text{----------- (17)}$$

This is the law of transformation of Christoffel symbols of the second kind and the occurrence of the last term in (17) shows that these functions are not components of a tensor. Or in the absence of the last term, it would have represented the law of transformation of a mixed tensor of order 3 (one contravariant and 2 covariant indices)

Note:

(i) Equation (14) is an obvious form of the law of transformation of the fundamental contravariant tensor. We know that

$$\bar{g}^{lk} = \frac{\partial \bar{x}^l}{\partial x^c} \frac{\partial \bar{x}^k}{\partial x^h} g^{ch}$$

or

$$\bar{g}^{lk} \frac{\partial x^d}{\partial \bar{x}^l} = \frac{\partial x^d}{\partial \bar{x}^l} \frac{\partial \bar{x}^l}{\partial x^c} \frac{\partial \bar{x}^k}{\partial x^h} g^{ch}$$

$$= \frac{\partial \bar{x}^k}{\partial x^h} \delta_c^d \, g^{ch}$$

$$= \frac{\partial \bar{x}^k}{\partial x^h} g^{dh}$$

(ii) In the absence of the last term in (13) and (17) they behave as the laws of transformations of respective tensors. This is the reason why the indices in the Christoffel symbols are called covariant type and contravariant types.

(iii) Equation (16), expressing second derivatives of x's with respect to $\bar{x}$'s in terms of Christoffel symbols and the first order derivatives, is as such important in further applications.

(iv) In every equation with tensor formalism, to check the correctness, balancing of the indices on both sides and in every term must be satisfied. This can be better understood if we look at equation (17).

On the left hand side of (17), all the indices are in the bar coordinate system and h appears as contravariant type and i, j as covariant type indices. On the right hand side, in the first term, a, b, d are repeated indices on which the summation is implied and there after the repeated indices loose their significance. Thus the left out indices are h contravariant and i and j covariant in the bar coordinate system. Similarly in the second term, d being the repeated index, we are left with h contravariant and i and j as covariant is the bar coordinate system. Thus the balancing is satisfied.

Q: If g_{ij} and a_{ij} are components of two symmetric covariant tensors and $\begin{Bmatrix} i \\ jk \end{Bmatrix}_g$ and $\begin{Bmatrix} i \\ jk \end{Bmatrix}_a$ are the corresponding Christoffel symbols of the second kind, prove that the quantities $\begin{Bmatrix} i \\ jk \end{Bmatrix}_g - \begin{Bmatrix} i \\ jk \end{Bmatrix}_a$ are the components of a mixed tensor, i being an index of contravariance and j and k indices of covariance.

Solution: It is given that g_{ij} and a_{ij} are two different fundamental tensors. So $\begin{Bmatrix} i \\ jk \end{Bmatrix}_g$ and $\begin{Bmatrix} i \\ jk \end{Bmatrix}_a$ are different Christoffel symbols with g_{ij} and a_{ij} fundamental tensors respectively. Writing the laws of transformation of these symbols, we have

$$\overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_g = \begin{Bmatrix} l \\ mn \end{Bmatrix}_g \frac{\partial x^m}{\partial \overline{x}^j} \frac{\partial x^n}{\partial \overline{x}^k} \frac{\partial \overline{x}^i}{\partial x^l} + \frac{\partial^2 x^l}{\partial \overline{x}^j \partial \overline{x}^k} \frac{\partial \overline{x}^i}{\partial x^l}$$

similarly,

$$\overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_a = \begin{Bmatrix} l \\ mn \end{Bmatrix}_a \frac{\partial x^m}{\partial \overline{x}^j} \frac{\partial x^n}{\partial \overline{x}^k} \frac{\partial \overline{x}^i}{\partial x^l} + \frac{\partial^2 x^l}{\partial \overline{x}^j \partial \overline{x}^k} \frac{\partial \overline{x}^i}{\partial x^l}$$

or $$\left(\overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_g - \overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_a \right) = \left(\begin{Bmatrix} l \\ mn \end{Bmatrix}_g - \begin{Bmatrix} l \\ mn \end{Bmatrix}_a \right) \frac{\partial x^m}{\partial \overline{x}^j} \frac{\partial x^n}{\partial \overline{x}^k} \frac{\partial \overline{x}^i}{\partial x^l}$$

Which reveals that the quantities $\overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_g - \overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}}_a$ are components of a third order mixed tensor, i being contravariant and j, k being covariant indices.

11.5 Transformation laws of velocity and acceleration vectors:

Q: Bring out the transformation laws for the velocity and acceleration vectors. Compare the components of these vectors with those in the cartesian coordinate system.

Solution: It x^i is the given coordinate system, then $\frac{dx^i}{dt}$ are the components of the velocity vector. Further,

$$u^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial \overline{x}^\alpha} \frac{d\overline{x}^\alpha}{dt} = \frac{\partial x^i}{\partial \overline{x}^\alpha} \overline{u}^\alpha \quad \text{-----------} \quad (1)$$

where $\overline{u}^\alpha$ are the velocity components in the $\overline{x}^i$ coordinate system. Thus

$u^i = \dfrac{\partial x^i}{\partial \bar{x}^\alpha}\,\bar{u}^\alpha$ shows the law of

transformation of a contravariant velocity vector. Again if u^i are the components of a velocity vector, then $a^i = \dfrac{du^i}{dt}$ are the components of acceleration vector.

So $\dfrac{du^i}{dt} = \dfrac{d}{dt}\left(\dfrac{\partial x^i}{\partial \bar{x}^\alpha}\bar{u}^\alpha\right)$ from (1)

$$= \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\,\frac{d\bar{x}^\beta}{dt}\,\bar{u}^\alpha + \frac{\partial x^i}{\partial \bar{x}^\alpha}\,\frac{d\bar{u}^\alpha}{dt} \quad \text{----------} \quad (2)$$

Substituting the corresponding expression for the second order derivatives in (2) in terms of Christoffel symbols and first order derivatives, we have

$$\frac{du^i}{dt} = \left(\overline{\begin{Bmatrix}\gamma \\ \alpha\beta\end{Bmatrix}}\frac{\partial x^i}{\partial \bar{x}^\gamma} - \begin{Bmatrix}i \\ ab\end{Bmatrix}\frac{\partial x^a}{\partial \bar{x}^\alpha}\frac{\partial x^b}{\partial \bar{x}^\beta}\right)\frac{d\bar{x}^\beta}{dt}\,\bar{u}^\alpha + \frac{\partial x^i}{\partial \bar{x}^\alpha}\,\frac{d\bar{u}^\alpha}{dt}$$

$$= \left(\overline{\begin{Bmatrix}\gamma \\ \alpha\beta\end{Bmatrix}}\bar{u}^\alpha\,\bar{u}^\beta\,\frac{\partial x^i}{\partial \bar{x}^\gamma}\right) - \begin{Bmatrix}i \\ ab\end{Bmatrix}\left(\frac{\partial x^a}{\partial \bar{x}^\alpha}\bar{u}^\alpha\right)\left(\frac{\partial x^b}{\partial \bar{x}^\beta}\bar{u}^\beta\right) + \frac{\partial x^i}{\partial \bar{x}^\alpha}\,\frac{d\bar{u}^\alpha}{dt}$$

or $$\left(\frac{du^i}{dt} + \begin{Bmatrix}i \\ ab\end{Bmatrix}u^a\,u^b\right) = \left(\overline{\begin{Bmatrix}\gamma \\ \alpha\beta\end{Bmatrix}}\bar{u}^\alpha\,\bar{u}^\beta\,\frac{d\bar{u}^\gamma}{dt}\right)\frac{\partial x^i}{\partial \bar{x}^\gamma}$$

(the repeated index α in the last term is replaced by γ)

if we put $a^i = \dfrac{du^i}{dt} + \begin{Bmatrix}i \\ ab\end{Bmatrix} u^a\,u^b$ ---------- (3)

the above equation reduces to

$$a^i = \frac{\partial x^i}{\partial \bar{x}^\gamma}\,\bar{a}^\gamma \quad \text{----------} \quad (4)$$

which shows the law of transformation of a contravariant acceleration vector.

In the Cartesian coordinate system, the equation (1) still holds good for the transformation as well as the components of a vector. However in the case of the components of the acceleration vector, they are represented by equation (3). In cartesian coordinate system, the fundamental tensor is nothing but a unit matrix and hence Christoffel symbols vanish. In such a case in equation (3), $a^i = \dfrac{du^i}{dt}$ only which is true in the Cartesian system.

11.6 Computation of Christoffel symbols:

Q: Find the Christoffel symbols corresponding to the line element

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta \,.\, d\phi^2$$

Solution: we have

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta \,.\, d\phi^2$$

where the g_{ij} tensor in this spherical polar coordinate system is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \text{ and } |\, g_{ij} \,| = g = r^4 \sin^{-2}\theta.$$

$$\text{Now } g^{11} = \frac{\text{minor of } g_{11}}{g} = \frac{r^4 \sin^2\theta}{r^4 \sin^2\theta} = 1$$

$$g^{22} = \frac{\text{minor of } g_{22}}{g} = \frac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \frac{1}{r^2} \text{ and}$$

$$g^{33} = \frac{\text{minor of } g_{33}}{g} = \frac{r^2}{r^4 \sin^2\theta} = \frac{1}{r^2 \sin^2\theta}$$ and the rest of the components of the

contravariant fundamental tensor vanish.

Let us put $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$

It is given that the non-vanishing components of g_{ij} are $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$.

In formulating the non-vanishing components of the Christoffel symbols, we follow accordingly.

a) Since $g_{ij} = 0$ for $i \neq j$, all Christoffel symbols with different indices vanish.

b) Since $g_{11} = 1$ (constant), all the Christoffel symbols with 1 index repeated, vanish.

c) When $g_{22} = r^2$ which is a function of x^1, all the Christoffel symbols vanish except index being 1. That is, the non-vanishing symbols are [1, 22] and [2, 22].

d) When we consider $g_{33} = r^2 \sin^2\theta$ which is a function of x^1 and x^2, all the symbols vanish except these with '3' index repeated and the third index as either 1 or 2. i.e., the non-vanishing symbols are [1, 33], [3, 13], [2, 33] and [3, 23].

Thus out of $3^3 = 27$ symbols, by symmetry property, they will be reduced to 18 independent symbols of which 12 are vanishing and the six non-vanishing elements are

$$[1,22] = \frac{1}{2}\left(0 + 0 - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial}{\partial r} \,.\, r^2 = -r.$$

$$[2,\, 12] = \frac{1}{2}\left(0 + \frac{\partial g_{22}}{\partial x^1} - 0\right) = \frac{1}{2} \,.\, 2r = r.$$

$$[1, 33] = \frac{1}{2}\left(0 + 0 - \frac{\partial g_{33}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial r^2 \sin^2\theta}{\partial r} = -r\sin^2\theta.$$

$$[3, 13] = \frac{1}{2}\left(0 + \frac{\partial g_{33}}{\partial x^1} - 0\right) = r\sin^2\theta.$$

$$[2,33] = \frac{1}{2}\left(0 + 0 - \frac{\partial g_{33}}{\partial x^2}\right) = -\frac{1}{2}\cdot\frac{\partial r^2 \sin^2\theta}{\partial\theta} = -r^2\sin\theta\cos\theta \quad \text{and}$$

$$[3,23] = \frac{1}{2}\left(\frac{\partial g_{33}}{\partial x^2}\right) = r^2\sin\theta\cos\theta.$$

The second kind of Christoffel symbols having non zero values are

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = g^{ij}\,[j, 22] = g^{11}[1,22] \text{ only} = 1\,(-r) = -r$$

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = g^{2j}\,[j, 12] = g^{22}\,[2,12] \text{ only} = \frac{1}{r^2}\,.\,r = \frac{1}{r}$$

$$\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = g^{1j}\,[j, 33] = g^{11}\,[1,33] \text{ only} = 1.(-r\sin^2\theta) = -r\sin^2\theta$$

$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = g^{3j}\,[j, 13] = g^{33}\,[3,13] \text{ only} = \frac{1}{r^2\sin^2\theta}\,.\,r\sin^2\theta = \frac{1}{r}$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = g^{2j}\,[j, 33] = g^{22}\,[2,33] \text{ only} = \frac{1}{r^2}\,.\,-r^2\sin\theta\cos\theta = -\sin\theta\cos\theta \quad \text{and}$$

$$\begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = g^{3j}\,[j, 23] = g^{33}\,[3,23] \text{ only} = \frac{1}{r^2\sin^2\theta}\,.\,r^2\sin\theta\cos\theta = \cot\theta.$$

<u>**Note:**</u> For a general line element in V_3, since all the elements g_{ij} are, in general, functions of the coordinates, all the $3^3 = 27$ elements will be present. In view of the symmetry nature of the symbols, the 18 independent elements can be represented by means of three upper triangular forms of matrices in the following way.

[1, 11]	[1, 12]	[1, 13]	[2, 11]	[2, 12]	[2, 13]	[3, 11]	[3, 12]	[3, 13]
	[1, 22]	[1, 23]		[2, 22]	[2, 23]		[3, 22]	[3, 23]
		[1, 33]			[2, 33]			[3, 33]

From these ordered forms it is easy or methodical to arrive at the non-vanishing symbols depending upon the nature of the fundamental tensor on hand.

Q: From the above problem, obtain the spherical polar components from the general definition of the acceleration vector

$$a^i = \frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt} \qquad (i, j, k = 1, 2, 3)$$

Solution: From the non zero three index symbols already derived in the above problem, the components of the acceleration vector are

$$a^1 = \frac{d^2x^1}{dt^2} + \begin{Bmatrix} 1 \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt} \qquad (j, k = 1, 2, 3)$$

or $$a^r = \frac{d^2r}{dt^2} + \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{d\theta}{dt}\frac{d\theta}{dt} + \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} \frac{d\phi}{dt}\frac{d\phi}{dt}$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2$$

Again $$a^\theta = \frac{d^2\theta}{dt^2} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \frac{dr}{dt}\frac{d\theta}{dt} + \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} \frac{d\phi}{dt}\frac{d\phi}{dt}$$

$$= \frac{d^2\theta}{dt^2} + \frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2$$

Lastly, $$a^\phi = \frac{d^2\phi}{dt^2} + \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} \frac{dr}{dt}\frac{d\phi}{dt} + \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} \frac{d\theta}{dt}\frac{d\phi}{dt}$$

$$= \frac{d^2\phi}{dt^2} + \frac{1}{r}\frac{dr}{dt}\frac{d\phi}{dt} + \cot\theta \,.\, \frac{d\theta}{dt}\frac{d\phi}{dt}$$

11.7 Summary:

The need for introduction of Christoffel symbols for a group of derivatives of the fundamental tensor is basically explained in the introduction. Two kinds of 3–index symbols (Christoffel) are defined along with their symmetry properties. Though there are two kinds of symbols, they are interrelated.

Transformation laws for the 3–index symbols have been derived and showed that they do not form the components of tensors. However, the notation is explained consistent with the contravariant and covariant indices.

A good example has been worked out on the transformation laws of velocity and acceleration vectors and a comparison is given with respect to the cartesian coordinate system.

Christoffel symbols have been calculated for a given line element. Though the calculations are cumbersome, a methodical procedure is explained. Cautionary notes is given wherever it is needed.

11.8 Key Terminology:

Christoffel symbol of 1st kind — second kind — vector and acceleration components — line element — contravariant type index — differentiation of a determinant.

11.9 Self – assessment questions:

1. Show with the usual notation

   (i) $\frac{\partial g_{ij}}{\partial x^k} = [i, jk] + [j, ik]$

   In what indices it is symmetric?

   (ii) $\begin{Bmatrix} i \\ ik \end{Bmatrix} = \frac{\partial}{\partial x^k} \log \sqrt{g}$

2. The line element in a two dimensional surface $\theta - \phi$ is given by $ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$ where R is a constant. Find all components of the metric tensor and the Christoffel symbols of first kind for this surface.
3. State and prove the transformation law for the Christoffel symbols of the first kind.
4. Show that $\frac{\partial g^{pq}}{\partial x^m} = -g^{p\alpha} \begin{Bmatrix} q \\ \alpha m \end{Bmatrix} - g^{q\alpha} \begin{Bmatrix} p \\ \alpha m \end{Bmatrix}$
5. What are Christoffel symbols? Show that they are not tensors. Describe the properties of Christoffel symbols.
6. Define the Christoffel's symbols of the first kind and the second kind and establish the relation between them. Explain the nature of indices and the symmetry properties of the symbols. Obtain the Christoffel symbols in the rectangular cartesian coordinate system.
7. Compute all the Christoffel symbols of first kind for the line element

   $ds^2 = e^{-\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + e^{\mu} dt^2$

8. Obtain the Christoffel symbols of both kinds

   for a space with the metric $ds^2 = f(u, v)du^2 + h(u, v)dv^2$

9. Express $\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}$ in Christoffel symbols of first kind

11.10 Reference Books:

1. C.E. Weatherburn 'Riemannian geometry and tensor calculus', Cambridge University Press, 1957

2. F.A. Hinchey, 'Vectors and tensors for engineers and scientists', Wiley Eastern Ltd., Delhi, 1976.

3. A.W. Joshi 'Matrices and Tensors in Physics',Wiley Eastern Ltd., Delhi, 1975.

4. B.S. Rajput 'Mathematical Physics', Pragati Prakashan, 1999.

Unit – III

Lesson – 12

COVARIANT DIFFERENTIATION

Objectives:

- To derive the components of velocity and acceleration vectors in the generalized coordinate system.
- To show that the coordinate derivatives of tensors are not tensors.
- To obtain the covariant derivatives of covariant vector, scalar invariant, contravariant vector, second order covariant tensor.
- To show that covariant derivatives of fundamental tensors and Kronecker delta vanish.
- Using covariant derivative, to being the expressions for grad, curl, div and Laplacian operations.

Structure:

12.1 Introduction:

In the earlier lesson – 2, it is learnt that the partial derivatives of a scalar field with respect to the coordinates are the components of a covariant vector. In general, however, differentiation of a tensor (except that of rank zero) with respect to the coordinates does not give a tensor. On the other hand, differentiation of a tensor with respect of any parameter other than the coordinates does not alter the nature and order of the tensor.

It is nevertheless possible to construct expressions containing the partial derivatives of a tensor with respect to the coordinates and those of the metric tensor transforming like a tensor. Such expressions will be defined and discussed in this chapter.

Once again we start with the already explained concept of velocity and acceleration vectors.

Q: Show that velocity and acceleration are contra-variant vectors.

Solution: **velocity:** If t denotes time and dx^i are the components of an infinitesimal contra-variant vector, then $\frac{dx^i}{dt}$ are the components of velocity vector. We know the transformation for dx^i as

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^j} dx^j$$

$$\text{or} \quad \frac{d\overline{x}^i}{dt} = \frac{\partial \overline{x}^i}{\partial x^j} \frac{dx^j}{dt}$$

$$\text{(i.e.),} \quad \overline{u}^i = \frac{\partial \overline{x}^i}{\partial x^j} u^j \quad \text{where } \overline{u}^i = \frac{d\overline{x}^i}{dt} \text{ (components of the velocity vector)}$$

This shows the quantities $\overline{u}^i$ transform like a contravariant vector and thus the velocity is a contravariant vector.

Acceleration: Since the velocity vector is represented by its contravariant components, we have

$$\overline{u}^\alpha = \frac{\partial \overline{x}^\alpha}{\partial x^i} u^i \quad \text{---------- (1)} \quad \text{where } u^i = \frac{dx^i}{dt}$$

Differentiating (1) w.r.t. time t, we get

$$\frac{d\overline{u}^\alpha}{dt} = \frac{\partial^2 \overline{x}^\alpha}{\partial x^i \partial x^j} u^i u^j + \frac{\partial \overline{x}^\alpha}{\partial x^i} \frac{du^i}{dt} \quad \text{----------(2)}$$

we know the expression for the second derivative of $\overline{x}^s$ w.r.t x'^s as (equation (16) in lesson 3 of this unit)

$$\frac{\partial^2 \overline{x}^\alpha}{\partial x^i \partial x^j} = \begin{Bmatrix} l \\ ij \end{Bmatrix} \frac{\partial \overline{x}^\alpha}{\partial x^l} - \overline{\begin{Bmatrix} \alpha \\ ab \end{Bmatrix}} \frac{\partial \overline{x}^a}{\partial x^i} \frac{\partial \overline{x}^b}{\partial x^j} \quad \text{---------- (3)}$$

substituting equation (3) in (2), it is seen that

$$\frac{d\bar{u}^{\alpha}}{dt} + \left\{\begin{matrix}\overline{\alpha} \\ ab\end{matrix}\right\} \frac{\partial \bar{x}^{a}}{\partial x^{i}} \frac{\partial \bar{x}^{b}}{\partial x^{j}} u^{i}u^{j} = \left\{\begin{matrix} l \\ ij\end{matrix}\right\} \frac{\partial \bar{x}^{\alpha}}{\partial x^{l}} u^{i}u^{j} + \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{du^{i}}{dt}$$

or
$$\left(\frac{d\bar{u}^{\alpha}}{dt} + \left\{\begin{matrix}\overline{\alpha} \\ ab\end{matrix}\right\}\bar{u}^{a}\bar{u}^{b}\right) = \left\{\begin{matrix} l \\ ij\end{matrix}\right\} \frac{\partial \bar{x}^{\alpha}}{\partial x^{l}} u^{i}u^{j} + \frac{\partial \bar{x}^{\alpha}}{\partial x^{l}} \frac{du^{l}}{dt}$$

$$= \left(\left\{\begin{matrix} l \\ ij\end{matrix}\right\}u^{i}u^{j} + \frac{du^{l}}{dt}\right) \frac{\partial \bar{x}^{\alpha}}{\partial x^{l}} \quad \text{----------} \quad (4)$$

in virtue of equation (1) and replacement of the dummy index i by l.

As it has been started with the time derivative of a velocity vector which should lead to the acceleration vector $\bar{a}^{\alpha} = \frac{d\bar{u}^{\alpha}}{dt} = \frac{d^{2}\bar{x}^{\alpha}}{dt^{2}}$ in the cartesian coordinate system, let us define the acceleration vector with its components in the generalized coordinate system as

$$\bar{a}^{\alpha} = \frac{d\bar{u}^{\alpha}}{dt} + \left\{\begin{matrix}\overline{\alpha} \\ ab\end{matrix}\right\} \bar{u}^{a}\bar{u}^{b} \quad \text{----------} \quad (5)$$

Thus equation (4) is simplified to

$$\bar{a}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{l}} a^{l} \quad \text{----------} \quad (6)$$

which shows that acceleration is a contravariant vector.

Note:

(i) dx^{i}, $\frac{dx^{i}}{dt}$ and $\frac{d^{2}x^{i}}{dt^{2}}$ are all the components of displacement vector, velocity vector and acceleration vector respectively and they are all contravariant vectors. So derivatives of the coordinates do not alter the nature and order of the tensor.

(ii) The components of the acceleration contravariant vector are given by (5)

$$\frac{d^{2}\bar{x}^{\alpha}}{dt^{2}} = \frac{d\bar{u}^{\alpha}}{dt} = \bar{a}^{\alpha} = \frac{d\bar{u}^{\alpha}}{dt} + \left\{\begin{matrix}\overline{\alpha} \\ ab\end{matrix}\right\} \bar{u}^{a}\,\bar{u}^{b}$$

In the Cartesian coordinate system, the fundamental (metric) tensor has all the components as constants and hence all the Christoffel symbols vanish. In such a case, the above equation is simply reduced to $\bar{a}^{\alpha} = \frac{d\bar{u}^{\alpha}}{dt}$.

Thus the definition of the acceleration vector in the generalized coordinate system as given by (5) is justified.

Q: Show that the coordinate derivatives of any tensor other than a scalar do not transform like the components of a tensor.

Solution: If $\phi(x^i)$ is a scalar invariant its derivatives with respect to the coordinates is given by

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \overline{\phi}}{\partial x^i} \qquad (\because \text{ scalar invariant})$$

$$= \frac{\partial \overline{\phi}}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^i}$$

or $$A_i = \frac{\partial \overline{x}^j}{\partial x^i} \overline{A}_j \quad \text{----------} \ (7)$$

Where $A_i = \dfrac{\partial \phi}{\partial x^i}$ = components of grand ϕ and equation (7) represents the law of transformation of a covariant vector. Thus the coordinate derivative of a tensor of order zero (scalar) is a covariant tensor of order one (vector).

Again consider a covariant vector transformation as

$$\overline{u}_\alpha = \frac{\partial x^i}{\partial \overline{x}^\alpha} u_i$$

Differentiating this w.r.t. $\overline{x}^\beta$, we get

$$\frac{\partial \overline{u}_\alpha}{\partial \overline{x}^\beta} = \frac{\partial^2 x^i}{\partial \overline{x}^\alpha \partial \overline{x}^\beta} u_i + \frac{\partial u_i}{\partial x^j} \frac{\partial x^i}{\partial \overline{x}^\alpha} \frac{\partial x^j}{\partial \overline{x}^\beta} \quad \text{-----------} \ (8)$$

The second term on the RHS of the above equation has a tensorial character, but the appearance of the first term shows that the functions $\dfrac{\partial \overline{u}_\alpha}{\partial \overline{x}^\beta}$ do not transform like the components of a second rank tensor.

12.2 Covariant derivative of a covariant vector:

The observations from equation (8) raise the important question of whether or not it is possible to add correction term to the partial derivatives $\dfrac{\partial \overline{u}_\alpha}{\partial \overline{x}^\beta}$ so that the result would be covariant tensor of order 2. By determining the appropriate correction terms, we define the so-called covariant derivative.

Let u_i and $\overline{u}_i$ be the components of a covariant vector in the coordinates x^i and $\overline{x}^i$ respectively. Differentiating the law of transformation

$$\overline{u}_i = \frac{\partial x^j}{\partial \overline{x}^i} u_j \quad \text{----------} \quad (9)$$

with respect to $\overline{x}^k$, we have

$$\frac{\partial \overline{u}_i}{\partial \overline{x}^k} = \frac{\partial u_j}{\partial x^a} \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial x^a}{\partial \overline{x}^k} + u_j \frac{\partial^2 x^j}{\partial \overline{x}^i \partial \overline{x}^k}$$

$$= \frac{\partial u_b}{\partial x^a} \frac{\partial x^b}{\partial \overline{x}^i} \frac{\partial x^a}{\partial \overline{x}^k} + u_j \left(\overline{\begin{Bmatrix} l \\ ik \end{Bmatrix}} \frac{\partial x^j}{\partial \overline{x}^l} - \begin{Bmatrix} j \\ ab \end{Bmatrix} \frac{\partial x^a}{\partial \overline{x}^i} \frac{\partial x^b}{\partial \overline{x}^k} \right)$$

(using similar expression as in (3))

$$= \frac{\partial u_b}{\partial x^a} \frac{\partial x^b}{\partial \overline{x}^i} \frac{\partial x^a}{\partial \overline{x}^k} + \overline{\begin{Bmatrix} l \\ ik \end{Bmatrix}} \overline{u}_l - \begin{Bmatrix} j \\ ab \end{Bmatrix} u_j \frac{\partial x^a}{\partial \overline{x}^i} \frac{\partial x^b}{\partial \overline{x}^k}$$

$$\text{or} \quad \left(\frac{\partial \overline{u}_i}{\partial \overline{x}^k} - \overline{\begin{Bmatrix} l \\ ik \end{Bmatrix}} \overline{u}_l \right) = \left(\frac{\partial u_a}{\partial x^b} - \begin{Bmatrix} j \\ ab \end{Bmatrix} u_j \right) \frac{\partial x^a}{\partial \overline{x}^i} \frac{\partial x^b}{\partial \overline{x}^k}$$

($\because$ a and b are dummy indices, they are interchanged in the first term.)

If we put $u_{i,k} = \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} l \\ ik \end{Bmatrix} u_l$ ---------- (10)

the above equation becomes

$$\overline{u}_{i,k} = u_{a,b} \frac{\partial x^a}{\partial \overline{x}^i} \frac{\partial x^b}{\partial \overline{x}^k}$$

showing that the quantities $u_{i,k}$ are components of a covariant tensor of the second order. This tensor is called the <u>covariant derivative</u> of the covariant vector u_i with respect to the fundamental tensor g_{ij}.

Covariant differentiation is indicated as above by a subscript preceded by a comma. Some authors use the notation as $u_{i;,k}$ instead of $u_{i,k}$.

12.3 Covariant derivative of a scalar invariant:

It is defined as the vector whose covariant components are its ordinary derivatives. Thus, if ϕ is scalar invariant,

$$\phi_{,i} = \frac{\partial \phi}{\partial x^i} \quad \text{----------} \quad (11)$$

The covariant derivative of this vector is denoted by $\phi_{,ij}$. This follows from equation (10) and it gives

$$(\phi_{,i})_{,j} = \frac{\partial(\phi,i)}{\partial x^j} - \begin{Bmatrix} l \\ ij \end{Bmatrix} (\phi_{,l})$$

$$\text{or } \phi_{,ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} - \begin{Bmatrix} l \\ ij \end{Bmatrix} \frac{\partial \phi}{\partial x^l}$$

RHS expression is symmetric in i and j

$$\therefore \phi_{,ij} = \phi_{,ji} \quad \text{----------} \quad (12)$$

In this case the order of the covariant differentiation is commutative. But this is the only case in which it is so.

12.4 Covariant derivative of a contravariant vector:

Let u^i and $\overline{u}^i$ be the components of a contravariant vector in two coordinate systems x^i and $\overline{x}^i$ respectively. Differentiating the law of transformation

$$u^i = \frac{\partial x^i}{\partial \overline{x}^j} \overline{u}^j \quad \text{----------} \quad (13)$$

w.r.t. x^k, we obtain

$$\frac{\partial u^i}{\partial x^k} = \frac{\partial \overline{x}^j}{\partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k} \frac{\partial x^i}{\partial \overline{x}^j} + \overline{u}^j \frac{\partial^2 x^i}{\partial \overline{x}^j \partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k}$$

By replacing the corresponding expression for $\frac{\partial^2 x^i}{\partial \overline{x}^j \partial \overline{x}^a}$,

$$\frac{\partial u^i}{\partial x^k} = \frac{\partial \overline{u}^j}{\partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k} \frac{\partial x^i}{\partial \overline{x}^j} + \overline{u}^j \left(\overline{\begin{Bmatrix} b \\ ja \end{Bmatrix}} \frac{\partial x^i}{\partial \overline{x}^b} - \begin{Bmatrix} i \\ hl \end{Bmatrix} \frac{\partial x^h}{\partial \overline{x}^j} \frac{\partial x^l}{\partial \overline{x}^a} \right) \frac{\partial \overline{x}^a}{\partial x^k}$$

$$\frac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ hl \end{Bmatrix} \left(\overline{u}^j \frac{\partial x^h}{\partial \overline{x}^j} \right) \frac{\partial x^l}{\partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k} = \frac{\partial \overline{u}^j}{\partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k} \frac{\partial x^i}{\partial \overline{x}^j} + \overline{u}^j \overline{\begin{Bmatrix} b \\ ja \end{Bmatrix}} \frac{\partial x^i}{\partial \overline{x}^b} \frac{\partial \overline{x}^a}{\partial x^k} \quad \text{----} \quad (14)$$

But $\overline{u}^j \frac{\partial x^h}{\partial \overline{x}^j} = u^h$ from (13)

$$\begin{Bmatrix} i \\ hl \end{Bmatrix} \frac{\partial x^l}{\partial \overline{x}^a} \frac{\partial \overline{x}^a}{\partial x^k} = \begin{Bmatrix} i \\ hl \end{Bmatrix} \delta^l_k = \begin{Bmatrix} i \\ hk \end{Bmatrix}$$

and $\dfrac{\partial \overline{u}^j}{\partial \overline{x}^a} \dfrac{\partial \overline{x}^a}{\partial x^k} \dfrac{\partial x^i}{\partial \overline{x}^j} = \dfrac{\partial \overline{u}^b}{\partial \overline{x}^a} \dfrac{\partial \overline{x}^a}{\partial x^k} \dfrac{\partial x^i}{\partial \overline{x}^b}$

($\because$ j is dummy and it is replaced by b for the sake of common factors.)

Therefore equation (14) becomes

$$\left(\frac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ hk \end{Bmatrix} u^h \right) = \left(\frac{\partial \overline{u}^b}{\partial \overline{x}^a} \overline{\begin{Bmatrix} b \\ ja \end{Bmatrix}} \overline{u}^j \right) \frac{\partial \overline{x}^a}{\partial x^k} \frac{\partial x^i}{\partial \overline{x}^b}$$

If we put $u^i{,}k = \dfrac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ hk \end{Bmatrix} u^h$ ---------- (15)

This equation becomes

$u^i{,}k = \overline{u}^j{,}a \dfrac{\partial x^i}{\partial \overline{x}^j} \dfrac{\partial \overline{x}^a}{\partial x^k}$ ---------- (16)

showing that $u^i{,}k$ are the components of a mixed tensor of the second order.

It is called the covariant derivative of the contravariant vector u^i with respect to the fundamental tensor.

Note: covariant differentiation of any tensor leads to another tensor with an additional covariant character. Thus

A^{ij}, k — 3rd order mixed tensor (2 contravariant and one covariant)

A_{ij}, k — 3rd order covariant tensor

A^i_j, k — 3rd order mixed tensor (one contravariant and 2 covariant).

There is no contravariant differentiation defined.

12.5 Covariant derivative of a second order covariant tensor:

We know the law of transformation of a second order covariant tensor as

$$\overline{A}_{ab} = \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} A_{ij} \quad \text{---------- (17)}$$

Differentiating w.r.t $\overline{x}^c$, we obtain

$$\frac{\partial \overline{A}_{ab}}{\partial \overline{x}^c} = \frac{\partial A_{ij}}{\partial x^h} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^h}{\partial \overline{x}^c} + A_{ij} \frac{\partial^2 x^i}{\partial \overline{x}^c \partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} + A_{ij} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial^2 x^j}{\partial \overline{x}^b \partial \overline{x}^c} \quad \text{----------- (18)}$$

$$= \frac{\partial A_{ij}}{\partial x^h} \cdot \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^h}{\partial \overline{x}^c} + A_{ij} \frac{\partial x^j}{\partial \overline{x}^b} \left(\overline{\begin{Bmatrix} h \\ a\,c \end{Bmatrix}} \frac{\partial x^i}{\partial \overline{x}^h} - \begin{Bmatrix} i \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial \overline{x}^a} \frac{\partial x^q}{\partial \overline{x}^c} \right)$$

$$+ A_{ij} \frac{\partial x^i}{\partial \overline{x}^a} \left(\overline{\begin{Bmatrix} h \\ b\,c \end{Bmatrix}} \frac{\partial x^j}{\partial \overline{x}^h} - \begin{Bmatrix} j \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial \overline{x}^b} \frac{\partial x^q}{\partial \overline{x}^c} \right) \quad \text{---------- (19)}$$

by virtue of equation (3)

Or

$$\frac{\partial \overline{A}_{ab}}{\partial \overline{x}^c} - \overline{\begin{Bmatrix} h \\ ac \end{Bmatrix}} \left(A_{ij} \frac{\partial x^i}{\partial \overline{x}^h} \frac{\partial x^j}{\partial \overline{x}^b} \right) - \overline{\begin{Bmatrix} h \\ bc \end{Bmatrix}} \left(A_{ij} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^h} \right)$$

$$= \frac{\partial A_{ij}}{\partial x^h} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^h}{\partial \overline{x}^c} - A_{ij} \begin{Bmatrix} i \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^q}{\partial \overline{x}^c} - A_{ij} \begin{Bmatrix} j \\ pq \end{Bmatrix} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^p}{\partial \overline{x}^b} \frac{\partial x^q}{\partial \overline{x}^c} \quad \text{----- (20)}$$

In the LHS expression, using equation (17)

$$A_{ij} \frac{\partial x^i}{\partial \overline{x}^h} \frac{\partial x^j}{\partial \overline{x}^b} = \overline{A}_{hb} \text{ and}$$

$$A_{ij} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^h} = \overline{A}_{ah}$$

In the RHS expression, interchange the repeated indices p and i, replace the repeated index q by h in the second term. Similarly interchange the repeated indices p and j and replace the repeated index q by h in the third term so that all the three partial derivatives occur as common factors. Thus equation (20) becomes

$$\left(\frac{\partial \overline{A}_{ab}}{\partial \overline{x}^c} - \overline{\begin{Bmatrix} h \\ ac \end{Bmatrix}} \overline{A}_{hb} - \overline{\begin{Bmatrix} h \\ bc \end{Bmatrix}} \overline{A}_{ah} \right) = \left(\frac{\partial A_{ij}}{\partial x^h} - \begin{Bmatrix} p \\ ih \end{Bmatrix} A_{pj} - \begin{Bmatrix} p \\ jh \end{Bmatrix} A_{ip} \right) \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^h}{\partial \overline{x}^c} \quad \text{------- (21)}$$

If then we put

$$A_{ij,h} = \frac{\partial A_{ij}}{\partial x^h} - \begin{Bmatrix} p \\ ih \end{Bmatrix} A_{pj} - \begin{Bmatrix} p \\ jh \end{Bmatrix} A_{ip} \quad \text{---------- (22)}$$

Equation (21) takes the form

$$\overline{A}_{ab,c} = A_{ij,h} \frac{\partial x^i}{\partial \overline{x}^a} \frac{\partial x^j}{\partial \overline{x}^b} \frac{\partial x^h}{\partial \overline{x}^c} \quad \text{----------- (23)}$$

which shows that the covariant derivative of a 2nd order covariant tensor is a third order covariant tensor. $A_{ij,h}$ as given in (22) is the covariant derivative of the second order covariant tensor w.r.t. the fundamental tensor.

Similarly it can be shown that the covariant derivatives of the tensors A^{ij} and A^i_j can be proved to be

$$A^{ij},k = \frac{\partial A^{ij}}{\partial x^k} + A^{ih}\begin{Bmatrix} j \\ hk \end{Bmatrix} + A^{hj}\begin{Bmatrix} i \\ hk \end{Bmatrix} \quad \text{----------} \quad (24)$$

and

$$A^i_{j,k} = \frac{\partial A^i_j}{\partial x^k} + A^h_j\begin{Bmatrix} i \\ hk \end{Bmatrix} - A^i_h\begin{Bmatrix} h \\ jk \end{Bmatrix} \quad \text{----------} \quad (25)$$

are third order mixed tensors with the indices as shown in equation (24) and (25).

Q: Show that the covariant derivatives of the tensors g_{ij}, g^{ij} and δ^i_j all vanish identically.

Solution: we know that

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{hj}\begin{Bmatrix} h \\ ik \end{Bmatrix} - g_{ih}\begin{Bmatrix} h \\ jk \end{Bmatrix}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - [j, ik] - [i, jk]$$

$$= [i, jk] + [j, ik] - [j, ik] - [i, jk] = 0$$

Similarly,

$$g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + g^{ih}\begin{Bmatrix} j \\ hk \end{Bmatrix} + g^{hj}\begin{Bmatrix} i \\ hk \end{Bmatrix}$$

But we know that $\frac{\partial g^{ij}}{\partial x^k} = - g^{ih}\begin{Bmatrix} j \\ hk \end{Bmatrix} - g^{hj}\begin{Bmatrix} i \\ hk \end{Bmatrix}$ ---------- (26)

Hence $g^{ij}_{,k} = 0$

Again $\delta^i_{j,k} = \frac{\partial \delta^i_j}{\partial x^k} + \delta^h_j\begin{Bmatrix} i \\ hk \end{Bmatrix} - \delta^i_h\begin{Bmatrix} h \\ jk \end{Bmatrix}$

$$= 0 + \begin{Bmatrix} i \\ jk \end{Bmatrix} - \begin{Bmatrix} i \\ jk \end{Bmatrix} = 0 \qquad (\because \delta^i_j \text{ is a kronecker delta})$$

A tensor $A^{ij....k}_{l\,m...n}$ of any order may be differentiated covariantly as already mentioned and its covariant derivative is given by

$$A^{ij....k}_{l\,m...n,\,p} = \frac{\partial}{\partial x^p} A^{ij....k}_{l\,m...n}$$

$$+A^{aj....k}_{l\,m...n}\begin{Bmatrix} i \\ ap \end{Bmatrix} + ... +A^{ij....a}_{l\,m...n}\begin{Bmatrix} k \\ ap \end{Bmatrix} - A^{ij....k}_{a\,m...n}\begin{Bmatrix} a \\ lp \end{Bmatrix} - ..... - A^{ij....k}_{l\,m...a}\begin{Bmatrix} a \\ np \end{Bmatrix} \quad \text{--------} \quad (27)$$

This is a tensor whose covariant order is greater by unity than that of $A^{aj....k}_{l\,m...n}$. The process of covariant differentiation can be repeated indefinitely. The covariant derivative of the first covariant derivative is called the second covariant derivative, and so on.

12.6 Curl of a vector:

Let u_i be the components of a covariant vector. Then its covariant derivative is given by

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \begin{Bmatrix} a \\ ij \end{Bmatrix} u_a.$$

then

$$u_{i,j} - u_{j,I} = \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} - \begin{Bmatrix} a \\ ij \end{Bmatrix} u_a + \begin{Bmatrix} a \\ ji \end{Bmatrix} u_a$$

$$= \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \quad \text{----------} \quad (28)$$

($\because$ the three index symbol is symmetric in i and j)

Since $u_{i,j}$ is a covariant tensor of order 2 and hence $u_{i,j}$ - $u_{j,i}$ should also be the same and it is called the <u>curl of the vector</u> $\hat{u}$

(i.e) curl $\hat{u} = u_{i,j} - u_{j,i} = \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}$ ---------- (29)

and it is an anti symmetric tensor.

Q: Prove that ordinary rule of differentiation of products also apply to the process of covariant differentiation.

Solution: Let us consider the product

$$C^i = A^i_j\, B^j$$

Then $C^i{}_{,k} = \frac{\partial C^i}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} C^\alpha$

$$= \frac{\partial}{\partial x^k} A^i_j\, B^j + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} C^\alpha$$

$$= \frac{\partial A^i_j}{\partial x^k} B^j + A^i_j \frac{\partial B^j}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} A^\alpha_j B^j$$

$$= \left[\frac{\partial A^i_j}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} A^\alpha_j \right] B^j + \frac{\partial B^j}{\partial x^k} A^i_j$$

$$= \left[\frac{\partial A^i_j}{\partial x^k} + \begin{Bmatrix} i \\ \alpha\ k \end{Bmatrix} A^\alpha_j - \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} A^i_\alpha \right] B^j + \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} A^i_\alpha B^j + \frac{\partial B^j}{\partial x^k} A^i_j$$

$$= A^i_{j,k} B^j + \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} A^i_j B^\alpha + \frac{\partial B^j}{\partial x^k} A^i_j$$

$$= A^i_{j,k} B^j + \left(\begin{Bmatrix} j \\ \alpha k \end{Bmatrix} B^\alpha + \frac{\partial B^j}{\partial x^k} \right) A^i_j = A^i_{j,k} B^j + B^j_{,k} A^i_j$$

which proves the result.

Q: A necessary and sufficient condition that the first covariant derivative of a covariant vector be symmetric is that the vector be a gradient.

Solution: If u_i denotes a covariant vector, then

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \begin{Bmatrix} k \\ ij \end{Bmatrix} u_k$$

$$u_{j,i} = \frac{\partial u_j}{\partial x^i} - \begin{Bmatrix} k \\ ji \end{Bmatrix} u_k$$

$$= \frac{\partial u_j}{\partial x^i} - \begin{Bmatrix} k \\ ij \end{Bmatrix} u_k$$ (due to the symmetry property of the Christoffel symbol).

$$\therefore\ u_{i,j} - u_{j,i} = \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} = \text{curl } \hat{u}$$

But given that $u_{i,j}$ is symmetric. (i.e) curl $\hat{u}$ = $u_{i,j} - u_{j,i} = 0$

so $\hat{u}$ must be a gradient of a scalar function. Conversely, if u_i are the components of the gradient of a scalar ϕ, then curl $\nabla\phi = 0$.

<u>12.7 Divergence of a vector:</u>

The divergence of a contravariant vector u^i may be defined as the contraction of its covariant derivative. It is thus a scalar invariant. We denote it briefly by div u^i.

Since $u^i_{,j} = \frac{\partial u_i}{\partial x^j} + u^h \begin{Bmatrix} i \\ hj \end{Bmatrix}$, it follows that

$$\text{Div } u^i = u^i_{,i} = \frac{\partial u^i}{\partial x^i} + u^h \begin{Bmatrix} i \\ hi \end{Bmatrix}$$

$$= \frac{\partial u^i}{\partial x^i} + u^h \frac{\partial}{\partial x^h} \log \sqrt{g} \qquad \text{(equation (7) of the previous lesson)}$$

$$= \frac{\partial u^i}{\partial x^i} + u^i \frac{\partial}{\partial x^i} \log \sqrt{g} \qquad \text{(since h is dummy)}$$

$$\text{Or div } u^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (u^i \sqrt{g}) \quad \text{----------} \quad (30)$$

12.8 Divergence of a tensor:

The divergence of a tensor is defined as its contracted covariant derivative with respect to the index of differentiation and any superscript.

Example: Consider the covariant derivative of a contravariant second rank tensor.

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} A^{\alpha j} + \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} A^{i\alpha}$$

Now div $(A^{ij}) = A^{ij}_{,j}$

$$= \frac{\partial A^{ij}}{\partial x^j} + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j} + \begin{Bmatrix} j \\ \alpha j \end{Bmatrix} A^{i\alpha}$$

$$= \frac{\partial A^{ij}}{\partial x^j} + \left(\frac{\partial}{\partial x^\alpha} \log \sqrt{g} \right) A^{i\alpha} + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j}$$

$$\text{since } \begin{Bmatrix} j \\ \alpha j \end{Bmatrix} = \frac{\partial}{\partial x^\alpha} \log \sqrt{g}$$

$$= \frac{\partial A^{ij}}{\partial x^j} + \left(\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} \right) A^{ij} + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (A^{ij} \sqrt{g}) + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j} \quad \text{----------} \quad (31)$$

The second term of (31), in view of the repeated indices α and j, can be written as

$$\begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j} = \begin{Bmatrix} i \\ j\alpha \end{Bmatrix} A^{j\alpha}$$

$$= \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{j\alpha} = - \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j} \text{ if } A^{\alpha j} \text{ is skew – symmetric.}$$

or $\begin{Bmatrix} i \\ \alpha j \end{Bmatrix} A^{\alpha j} = 0$ if $A^{\alpha j}$ is skew-symmetric .

so if A^{ij} is the skew – symmetric tensor, then equation (31) becomes

$$\text{div}\,(A^{ij}) = A^{ij}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(A^{ij}\sqrt{g}) \quad \text{----------} \quad (32)$$

12.9 Laplacian of a scalar invariant:

We know that the gradient of a scalar field ϕ, is a covariant vector A_i given by

$$A_i = \frac{\partial \phi}{\partial x^i} = \phi,_i$$

Its contravariant components can be represented as

$$A^i = g^{ij} A_j = g^{ij}\frac{\partial \phi}{\partial x^j} = g^{ij}\phi,_j$$

Div grad $\phi = \hat{\nabla}\cdot\hat{\nabla}\phi = \nabla^2\phi$ where ∇^2 is Laplacian operator

$$= A^i,i$$

$$= (g^{ij}\phi,_j),i = g^{ij},_i\,\phi,_j + g^{ij}\phi,_{ji}$$

$$= 0 + g^{ij}\phi,_{ij} \quad \text{----------} \quad (33)$$

since the covariant derivative of the fundamental tensor vanishes and $\phi,_{ij}$ is symmetric.

$$\text{So } \nabla^2\phi = g^{ij}\left[\frac{\partial \phi,_i}{\partial x^j} - \begin{Bmatrix} k \\ ij \end{Bmatrix}\phi,k\right] = g^{ij}\left[\frac{\partial^2\phi}{\partial x^i \partial x^j} - \frac{\partial \phi}{\partial x^k}\begin{Bmatrix} k \\ ij \end{Bmatrix}\right] \quad \text{----------} \quad (34)$$

Other form is

$$\nabla^2\phi = \text{div grad }\phi = A^i,_i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\,.\,A^i)$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial \phi}{\partial x^j}\right) \quad \text{----------} \quad (35)$$

in which case, the Laplacian operator can be written as

$$\nabla^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial x^j}\right) \quad \text{----------} \quad (36)$$

12.10 Summary:

A need for covariant differentiation other than ordinary differentiation is explained in the introduction.

Velocity and acceleration vectors are once again treated. Their representations in the generalized coordinate system and cartesian coordinate system are thoroughly distinguished.

The concept of a covariant derivative of a covariant vector via ordinary differentiation is introduced. This has been extended to all other types of tensors.

Special cases of the covariant derivatives of a scalar function ϕ, g_{ij}, g^{ij} and δ^i_j are treated. Ordinary rules of differentiation of products will hold good even in covariant differentiation.

Curl, div and Laplacian in vector analysis are once again explained with their expressions in terms of covariant derivatives.

12.11 Key Terminology:

Covariant derivative — curl — direvgence — Laplacian operators.

12.12 Self – assessment questions:

1. Find the covariant derivative of the gradient of a scalar function ϕ and show that it is a symmetric covariant tensor of order two.
2. Find the expression for the divergence of a symmetric contravariant second rank tensor.
3. Prove that $A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}(A^{ij}\sqrt{g}) + A^{jk}\begin{Bmatrix} i \\ jk \end{Bmatrix}$ the last term vanishing if A^{jk} is skew-symmetric. Also show that

$$A^j_{i,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}(A^j_i\sqrt{g}) - A^j_k\begin{Bmatrix} k \\ ij \end{Bmatrix}$$

4. If A_{ij} is the curl of a covariant vector, prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$ and that this is equivalent to

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0$$

5. A fluid in motion in a plane has the velocity vector field given by $u^i = (x^2, y^2)$ in cartesian coordinates. Find the covariant derivative of the vector field in polar coordinates.

12.13 Reference Books:

1. C.E. Weatherburn ‘Riemannian geometry and tensor calculus’, Cambridge University Press, 1957.

2. F.A. Hinchey, ‘Vectors and tensors for engineers and scientists’, Wiley Eastern Ltd., Delhi, 1976.

3. A.W. Joshi, ‘Matrices and tensors in Physics’, Wiley Eastern Ltd., Delhi, 1975.

Unit - IV

Lesson - 13

LAPLACE TRANSFORMS

Objective of the lesson :

> To define Laplace transform from Integral transform.
> To define the properties of Laplace transforms with examples.
> To obtain Laplace transforms of some special functions.
> To evaluate certain integrals using Laplace transform techniques.
> To provide table of Laplace transforms for quick reference.

Structure of the Lesson :

13.1 Introduction

The Laplace transformation is a powerful method for solving linear differential equations with constant coefficients arising in engineering mathematics. In the first step the differential equation is transformed into an algebraic equation. Nextly, the algebraic equation is solved with algebraic manipulations. Finally, the solution of the algebraic equation is transformed back in such a way it becomes the solution of the original differential equation.

Apart from the advantage of reducing the differential equation into an algebraic equation, another advantage is that it takes care of initial conditions without the necessity of first determining the general solution and then obtaining from it a particular solution. Also, when applying the classical method to a non homogeneous equation, we must first solve the corresponding homogeneous equation, while the Laplace transformation immediately yields the solution of the non homogeneous equation.

These transformation techniques are also useful in solving the boundary value problems and in the evaluation of certain integrals with much ease.

13.2 Definition of Laplace transforms :

Linear integral transformations of functions F(t) defined on a finite or infinite interval $a<t<b$ are particularly useful in solving problems in differential equations. Let K (t,s) denote

some prescribed function of the variable t and a parameter s. A general linear integral transformation of functions F(t) with respect to the kernel function K (t,s) is represented by the equation

$$T\{F(t)\} = \int_a^b K(t,s)\,F(t)\,dt \quad \text{-------- (1)}$$
$$= f(s)$$

It represents a function f(s), the image or transform of the function F(t).

If a function F(t) defined for all positive values of the variable t, is multiplied by the Kernel function e^{-st} and integrated with respect to t from zero to infinity, a new function f(s) is obtained. That is,

$$\left.\begin{aligned}\int_0^\infty e^{-st}F(t)\,dt &= f(s)\\ &= L\{F(t)\}\end{aligned}\right\} \quad ------ \;(2)$$

This operation on a function F(t) is called the Laplace transformation or Laplace transform of F(t). The new function f(s) is called the Laplace transform of F(t). Hereafter, we shall denote the original function by a capital letter and its transform by the same letter in lower case.

Ex (1) : Find the Laplace transform of the following functions :

(i) F(t) = 1 (ii) F(t) = t (iii) F(t) = e^{kt}

Sol :- (i) As per definition

$$L\{1\} = \int_0^\infty e^{-st}.1.\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s} \qquad s > 0$$

(ii) $$L\{t\} = \int_0^\infty e^{-st}.t.\,dt = t.\left.\frac{e^{-st}}{-s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}.1.dt$$

$$= 0 + \frac{1}{s}.\frac{1}{s} = \frac{1}{s^2} \qquad \text{using (i)}$$

(iii) $$L\{e^{kt}\} = \int_0^\infty e^{-st}.e^{kt}.\,dt = \int_0^\infty e^{-(s-k).t}.dt$$

$$= \frac{1}{s-k} \qquad \text{using (i)}$$

13.3 Some properties of Laplace transforms :

13.3.1 Linearity Property :

A Laplace transform L{F(t)} is said to be linear if for every pair of functions $F_1(t)$ and $F_2(t)$ and for every pair of constants C_1 and C_2, we have

$$L\{C_1 F_1(t)+C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$

$$= C_1 f_1(s) + C_2 f_2(s)$$

where $f_1(s)$ and $f_2(s)$ are Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively

Proof : We have $L\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)dt$

and $L\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)dt$

so that $L\{C_1F_1(t)\} = C_1 f_1(s) = \int_0^\infty e^{-st} C_1 F_1(t)dt = C_1 L\{F_1(t)\}$

and $L\{C_2F_2(t)\} = C_2 f_2(s) = \int_0^\infty e^{-st} C_2 F_2(t)dt = C_2 L\{F_2(t)\}$

$\therefore L\{C_1 F_1(t)+C_2F_2(t)\} = \int_0^\infty e^{-st} \{C_1 F_1(t)+C_2 F_2(t)\}dt$ by definition

$$= \int_0^\infty e^{-st} C_1 F_1(t)dt + \int_0^\infty e^{-st} C_2 F_2(t)dt$$

$$= C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$

$$= C_1 f_1(s) + C_2 f_2(s) \quad \text{........ (3)}$$

The result may be generalized for any number of functions and for the same number of arbitrary constants i.e.,

$$L\left\{\sum_{r=1}^{n} C_r F_r(t)\right\} = \sum_{r=1}^{n} C_r L\{F_r(t)\} \quad \text{......... (4)}$$

<u>**Ex (2) :**</u> Find the Laplace Transform of $4e^{5t}+6t^3-4\cos 3t+3\sin 4t.$

<u>**Soln :**</u> Applying the linearity property, we have

$$L\{4e^{5t}+6t^3-4\cos 3t+3\sin 4t\}$$

$$= 4L\{e^{5t}\} + 6L\{t^3\} - 4L\{\cos 3t\} + 3L\{\sin 4t\}$$

$$= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 4\left(\frac{s}{s^2+9}\right) + 3\left(\frac{4}{s^2+16}\right)$$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{4s}{s^2+9} + \frac{12}{s^2+16}$$

13.3.2 First Translation (or Shifting) Property :

If f(s) be the Laplace transform of F(t), then the Laplace transform of $e^{at}F(t)$ is f(s-a), where a is any real or complex number i.e., if

$L\{F(t)\} = f(s)$, then $L\{e^{-st}F(t)\} = f(s-a)$

Proof : Given, $L\{F(t)\} = \int_0^{\infty} e^{-st}F(t)\,dt = f(s)$

$$\therefore L\{e^{at}F(t)\} = \int_0^{\infty} e^{-st}e^{at}F(t)\,dt$$

$$= \int_0^{\infty} e^{-(s-a)t}F(t)\,dt$$

$$= \int_0^{\infty} e^{-ut}F(t)\,dt \quad \text{by putting } u = s-a$$

= f(u)

= f(s-a) .... (5)

Ex (3) : Find the Laplace transform of $e^{-2t}\sin 3t$

Soln : We have $L\{\sin 3t\} = \frac{3}{s^2+9}$

$$\therefore L\{e^{-2t}\sin 3t\} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13} .$$

13.3.3 Second Translation (or Shifting) Property :

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

Then $L\{G(t)\} = e^{-as} f(s)$.

Proof : We have $L\{G(t)\} = \int_0^{\infty} e^{-st}G(t)\,dt$

$$= \int_0^a e^{-st}\, G(t)\, dt + \int_a^\infty e^{-st}\, G(t)\, dt$$

$$= \int_0^a e^{-st}.0\ dt + \int_a^\infty e^{-st}\, F(t-a)\ dt$$

$$= \int_0^\infty e^{-st}\, F(t-a)\ dt$$

$$= \int_0^\infty e^{-s(u+a)}\, F(u)\, du\,,$$ by taking u = t-a i.e., du = dt.

when t = a, u = 0 and when t = ∞, u = ∞.

$$= e^{-sa} \int_a^\infty e^{-su}\, F(u)\, du$$

$$= e^{-sa}\, f(s) \qquad \text{................} \quad (6)$$

Ex (4) : Find the Laplace transform of F(t), where

$$F(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0\ , & t < \dfrac{2\pi}{3} \end{cases}$$

Soln : We have, $L\{F(t)\} = \int_0^\infty e^{-st}\ F(t)$

$$= \int_0^{2\pi/3} e^{-st}.0.\ dt + \int_{2\pi/3}^\infty e^{-st}\ \cos\left(t - \frac{2\pi}{3}\right)\ dt$$

$$= \int_0^\infty \exp\left[-s\left(u + \frac{2\pi}{3}\right)\right]\cos u\ du$$ by taking $u = t - \dfrac{2\pi}{3}$

$$= e^{-\frac{2\pi s}{3}}\ L\{\cos u\}$$

$$= e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}$$

13.3.4 The Change of Scale of Property :

If $L\{F(t)\} = f(s)$, then $L\{F(at)\}\ \dfrac{1}{a}\, f\left(\dfrac{s}{a}\right)$ .... (7)

Proof : We have $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$

$\therefore \quad L\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt$ (on replacing t by at)

$= \int_0^\infty e^{-su/a} F(u) \frac{du}{a}$ by taking at = u

$= \frac{1}{a} \int_0^\infty e^{-pu} F(u)\, du$ where $p = \frac{s}{a}$.

$= \frac{1}{a} \int_0^\infty e^{-pt} F(t)\, dt$ (replacing u by t)

$= \frac{1}{a} f(p)$

$= \frac{1}{a} f\left(\frac{s}{a}\right) \quad \because p = \frac{s}{a}$.

Ex (5) : Find the Laplace transform of $\cos 5t$.

Soln : We have, $L\{\cos t\} = \frac{s}{s^2+1}, \quad s > 0$

$$\therefore \quad L\{\cos 5t\} = \frac{1}{5} \cdot \frac{s/5}{(s/5)^2+1} = \frac{s}{s^2+25}$$

13.3.5 Laplace Transform of Derivatives :

If F(t) is continuous for $t \geq 0$ and of exponential order as $t \to \infty$ while $F'(t)$ is sectionally continuous i.e., $F'(t)$ is of class A for $t \geq 0$, and if $L\{F(t)\} = f(s)$, then $L\{F'(t)\} = s f(s) - F(0)$.

In general if $L\{F(t)\} = f(s)$ and $F(t), F'(t), F''(t), \ldots\ldots F^{(n-1)}(t)$ are continuous for $t \geq 0$ and of exponential order as $t \to \infty$ while $F^{(n)}(t)$ is sectioally continuous for $t \geq 0$, then

$$L\{F^{(n)}(t)\} = s^n f(s) - \sum_{r=0}^{n-1} s^{n-1-r} F^{(r)}(0)$$

$$= s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) \ldots\ldots - s F^{(n-2)}(0) - F^{(n-1)}(0)$$

Proof : Since $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$

$$\therefore \quad L\{F'(t)\} = \int_0^\infty e^{-st}\, F'(t)\, dt = \left[e^{-st}\, F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\, dt, \quad \text{integrating by parts}$$

$$= 0 - F(0) + s f(s)$$

$$= s f(s) - F(0) \qquad \text{........... (8)}$$

Applying the result (8), we have

$$L\{F''(t)\} = s L\{F'(t)\} - F'(0)$$

$$= s\{s f(s) - F(0)\} - F'(0) \qquad \text{by (8)}$$

$$= s^2 f(s) - s F(0) - F'(0) \qquad \text{........... (9)}$$

Similarly $\quad L\{F'''(t)\} = s^2 f(s) - s^2 F(0) - s F'(0) - F''(0) \qquad$ .... (10)

Generalizing it, we find

$$\left.\begin{aligned} L\{F^{(n)}(t)\} &= s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) \,.......... - s F^{(n-2)}(0) \\ &\qquad - F^{(n-1)}(0) \\ &= s^n f(s) - \sum_{r=0}^{n-1} s^{n-1-r}\; F^{(r)}(0) \end{aligned}\right\} \text{........... (11)}$$

Ex (6) : Find the Laplace transform of $F'(t)$ when $F(t) = e^{3t}$

Soln : Given $F(t) = e^{3t}$, $\therefore F(0) = 1$ and $F'(t) = 3e^{3t}$

As such $L\{e^{3t}\} = s L\{e^{3t}\} - 1$ by (8) above

$$= \frac{s}{s-3} - 1 = \frac{3}{s-3}$$

Aliter. $\quad L\{F'(t)\} = L\{3e^{3t}\} = \dfrac{3}{s-3}$.

13.3.6 Derivatives of Laplace Transforms :

If the function F(t) is sectionally continuous for $t \geq 0$ and if $L\{F(t)\} = f(s)$, then $f'(s) = L\{-t F(t)\}$.

Proof : We have $\quad f(s) = \int_0^\infty e^{-st}\; F(t)\; dt$

Differentiating either side w.r.t. 's' we get

$$f'(s) = \int_0^\infty (-t)\, e^{-st}\, F(t)\, dt = \int_0^\infty e^{-st}\, \{-t\, F(t)\}\, dt$$

$$= L\{-t F(t)\} \qquad \text{.............. (12)}$$

In general if F(t) is sectionally continuous for $t \geq 0$ and if

$L\{F(t)\} = f(s)$, then

$$f^{(n)}(s) = L\{(-t)^n F(t)\} \qquad \text{.............. (13)}$$

where $f^{(n)}(s) = \frac{d^n}{ds^n} f(s)$ for all integral values of n.

We may state it as

$$L\{t^n F(t)\} = (-1)^n f^n(s) = (-1)^n \frac{d^n}{ds^n} f(s) \qquad \text{............... (14)}$$

Ex (7) : Find the Laplace transform of $t^3 e^t$.

Soln: Since $L\{e^t\} = f(s) = \frac{1}{s-1}$

$$\therefore \quad L\{t^3 e^t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s-1}\right) = (-1)^3 = \frac{(-1)(-2)(-3)}{(s-1)^4}$$

$$= \frac{6}{(s-1)^4}$$

13.3.7 Laplace Transform of Integrals :

(i) If $L\{F(t)\} = f(s)$, then $L\left\{\int_0^t F(u)\, du\right\} = \frac{f(s)}{s}$ .... (15)

Proof : Let $G(t) = \int_0^t F(u)\, du$

Then $G'(t) = \frac{d}{dt}\left[\int_0^t F(u)\, du\right] = F(t)$ and $G(0) = \int_0^0 F(u)\, du = 0$.

Applying the property [13.3.5], we have

$$L\{G'(t)\} = s L\{G(t)\} - G(0)$$

i.e., $L\{F(t)\} = s L\{G(t)\} - 0$ or $f(s) = s L\left\{\int_0^t F(u)\, du\right\}$

i.e., $L\left\{\int_0^t F(u)\, du\right\} = \dfrac{f(s)}{s}$

(ii) If $L\{F(t)\} = f(s)$ then $L\left\{\dfrac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du$ .... (16)

Proof : Let $G(t) = \dfrac{F(t)}{t}$, so that $F(t) = t\,G(t)$

$\therefore\ L\{F(t)\} = L\{t\,G(t)\}$ (on taking Laplace transform)

$= (-1)\dfrac{d}{ds} L\{G(t)\}$ by property [13.3.6]

i.e., $-f(s) = \dfrac{d}{ds} L\{G(t)\}$

Integrating both sides with regard to s, we get

$$-\int_\infty^s f(s)\, ds = L\{G(t)\}$$

i.e., $L\{G(t)\} = \int_s^\infty f(u)\, du$, on the assumption that $\lim\limits_{s\to\infty} L\{G(s)\} \to 0$.

Ex (8) : Find the Laplace transform of $\int_0^\infty \dfrac{\sin t}{t}\, dt$.

Soln : We have $L\{\sin t\} = \dfrac{1}{s^2+1} = f(s)$ (say)

and $L\left\{\dfrac{\sin t}{t}\right\} = \int_s^\infty \dfrac{du}{u^2+1}$ by (16), $\because f(u) = \dfrac{1}{u^2+1}$

$$= \left[\tan^{-1} u\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s \qquad \because \tan^{-1} s + \cot^{-1} s = \frac{\pi}{2}$$

$$= \tan^{-1}\frac{1}{s}$$

Hence by (15) of property [13.3.7],

$$L\left\{\int_0^\infty \frac{\sin t}{t}\, dt\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

13.3.8 Periodic Functions :

If F(t) is a periodic function with period T>0, so that

$$F(t+T) = F(t), \quad \text{then } L\{F(t)\} = \frac{\int_0^T e^{-st}\, F(t)\, dt}{1-e^{-sT}} \qquad \text{............ (17)}$$

Proof : We have $L\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\, dt$

$$= \int_0^T e^{-st}\, F(t)\, dt + \int_T^{2T} e^{-st}\, F(t)\, dt + \int_{2T}^{3T} e^{-st}\, F(t)\, dt + \text{...........}$$

$$= \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}\, F(t)\, dt$$

If we put t=u+nT, then F(u+nT) = F(u) $\because$ F(t+T) = F(t) (given)

Thus,

$$L\{F(t)\} = \sum_{n=0}^{\infty}\int_0^T e^{-s(u+nT)}\, F(u)\, du$$

or $$L\{F(t)\} = \sum_{n=0}^{\infty} e^{-snT}\int_0^T e^{-su}\, F(u)\, du$$

$$= \left(1+e^{-sT}+e^{-2sT}+......\right)\int_0^T e^{-su}\, F(u)\, du$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-su}\, F(u)\, du \qquad \because (1-e^{-sT})^{-1} = 1+e^{-sT}+e^{-2sT}+.... \text{ when } \left|e^{-st}\right|<1$$

$$= \frac{\int_0^T e^{-st}\, F(t)\, dt}{1-e^{-sT}} \quad \text{(replacing u by t)}$$

Ex (9) : Find the Laplace transform of F(t) when F(t) is a periodic function with period 2π, such that

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

Soln : We have $L\{F(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}\, F(t)\, dt$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt + \int_{\pi}^{2\pi} e^{-st}.0\ dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1}\ (-s \sin t - \cos t) \right]_0^{\pi} + 0$$

$$\because \int e^{ax} \operatorname{Sin} bx\ dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$$= \frac{1}{1-e^{-2\pi s}}\ \frac{e^{-\pi s}+1}{s^2+1}$$

$$= \frac{1}{(1-e^{-\pi s})\,(1+e^{-\pi s})} \cdot \frac{1+e^{-\pi s}}{1+s^2}$$

$$= \frac{1}{(1-e^{-\pi s})\,(1+s^2)}\ .$$

13.3.9 Initial Value Theorem :

If $L\{F(t)\} = f(s)$ then $\lim_{t\to 0} F(t) = \lim_{s\to\infty} s f(s)$ .... (18)

Proof : We have $L\{F'(t)\} = s f(s) - F(0)$ by property [13.3.5]

i.e., $\int_0^{\infty} e^{-st}\, F'(t)\, dt = s f(s) - F(0)$

Taking the limit as $s \to \infty$,

$$\lim_{s\to\infty} \int_0^{\infty} e^{-st}\, F'(t)\, dt = \lim_{s\to\infty} s f(s) - F(0)$$

or $$\lim_{s\to\infty} s f(s) = F(0) + \int_0^{\infty} \left(\lim_{s\to\infty} e^{-st} \right) F'(t)\, dt$$

$$= F(0) + 0 \qquad \because \lim_{s\to\infty} e^{-st} = 0$$

$$= \lim_{t\to 0} F(t)$$

Ex (10) : Verify the initial value theorem for the function

$F(t) = e^{-3t}$

Soln : $f(s) = L\{F(t)\} = L\{e^{-3t}\} = \dfrac{1}{s+3}$

Now $\lim\limits_{t\to 0} F(t) = \lim\limits_{t\to 0} e^{-3t} = 1$

and $\lim\limits_{s\to \infty} F(t) = \lim\limits_{s\to \infty} \dfrac{1}{s+3} = 1$.

Hence $\lim\limits_{t\to 0} F(t) = \lim\limits_{s\to \infty} s f(s)$.

13.3.10 Final Value Theorem :

If $L\{F(t)\} = f(s)$, then $\lim\limits_{s\to 0} s f(s) = \lim\limits_{t\to \infty} F(t)$ .... (19)

Proof : We have $L\{F'(t)\} = s f(s) - F(0)$ by Prop. [13.3.5]

i.e., $\int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0)$

Taking the limit as $s \to 0$

$$\lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s\to 0} s f(s) - F(0)$$

or $$\lim_{s\to 0} s f(s) = F(0) + \lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\, dt$$

$$= F(0) + \int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) F'(t)\, dt$$

$$= F(0) + \int_0^\infty 1.\ F'(t)\, dt$$

$$= F(0) + \int_0^\infty \frac{d}{dt} F(t)\, dt$$

$$= F(0) + [F(t)]_0^\infty$$

$$= F(0) + \lim_{t\to\infty} F(t) - F(0)$$

$$= \lim_{t\to\infty} F(t).$$

Ex (11) : Verify the Final value theorem for the function

$F(t) = e^{-2t}$.

Soln : We have $F(t) = e^{-2t}$ so that $f(s) = L\{F(t)\} = L\{e^{-2t}\} = \dfrac{1}{s+2}$

$$\therefore \lim_{t \to \infty} F(t) = \lim_{t \to 0} e^{-2t} = 0$$

and $\lim_{s \to 0} s f(s) = \lim_{s \to 0} \dfrac{s}{s+2} = 0$

Hence $\lim_{t \to \infty} F(t) = \lim_{s \to \infty} s f(s)$.

13.3.11 Behaviour of *f(s)* as $s \to 0$ and $s \to \infty$:

We have $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$

when $s \to 0$, $f(0) = \int_0^\infty F(t)\, dt$ .... (20)

and when $s \to \infty$, $\lim_{s \to \infty} f(s) = \int_0^\infty 0 \,.\, F(t)\, dt = 0$ .... (21)

13.4 THE LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

[13.4.1] The Gamma function. Euler's Gamma function is defined as

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx\,, \qquad n > 0$$

Its important properties have already been discussed in an earlier lesson, but a few of them are mentioned here.

$$\Gamma(n+1) = n\,\Gamma n \quad \text{for } n > 0 \qquad \text{and} \qquad \Gamma n = \frac{\Gamma(n+1)}{n} \quad \text{for } n < 0$$

$\Gamma(n+1) = n!$ for positive integral values of n.

$$\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}\,, \qquad 0 < n < 1,\ \Gamma\frac{1}{2} = \sqrt{\pi}$$

Now we have $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$

Put $st = u$ i.e., $t = \dfrac{u}{s}$ and $dt = \dfrac{du}{s}$

$$\therefore \quad L\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \text{.......... (22)}$$

(by the definition of Gamma function)

If we now put $n = -\frac{1}{2}$,

$$L\{t^{-1/2}\} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}} \qquad \text{.............. (23)}$$

13.4.2 Bessel Functions :

Bessel function of order n is defined as

$$J_n(t) = \frac{t}{2^n \Gamma(n+1)} \left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4\,(2n+2)\,(2n+4)} \;............\right\}$$

which satisfies Bessel's differential equation

$$y''(t) + \frac{1}{t} y'(t) + \left(1 - \frac{n^2}{t^2}\right) y(t) = 0$$

or $\quad t^2 J''_n(t) + t J'_n(t) + (t^2 - n^2) J_n(t) = 0$

Some important properties are :

$J_{-n}(t) = (-1)^n J_n(t)$, n being positive integral

$J_{n+1}(it) = i^{-n} J_n(t)$, J_n being modified Bessel function of order n.

$$J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$$

$\frac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$ which becomes $J'_0(t) = -J_1(t)$ for n=0

$$e^{\frac{t}{2}\left(x - \frac{1}{x}\right)} = \sum_{n=-\infty}^{\infty} J_n(t)\, x^n$$

known as generating function for the Bessel functions.

$J_0(t)$ is called Bessel function of order zero and has for its expansion

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + ...$$

$$\therefore\ L\{J_0(t)\} = L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2.4^2}\right\} - L\left\{\frac{t^6}{2^2.4^2.6^2}\right\} + ...$$

$$= \frac{1}{s} - \frac{2\,!}{2^2 s^3} + \frac{4\,!}{2^2.4^2 s^5} ....$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) ....\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{s^2+1}} \qquad .... (24)$$

Similarly $L\{J_1(t)\} = 1 - s\Big/\sqrt{s^2+1}$ .... (25)

Aliter. $J_0(t)$ satisfies the equation

$$t\,J_0''(t) + J_0'(t) + t\,J_0(t) = 0$$

$$\therefore\ L\{t\,J_0''(t) + J_0'(t) + t\,J_0(t)\} = 0$$

Taking $L\{J_0(t)\} = f(s)$ and using properties [13.3.5] and [13.3.6].

$$-\frac{d}{ds}\{s^2 f(s) - s F(0) - F'(0)\} + \{s f(s) - F(0)\} - \frac{d}{ds} f(s) = 0$$

where $F(t) = J_0(t)$ gives $F(0) = 1$ and $F'(0) = 0$

$$\therefore\ -2s\,f(s) - s^2\,f'(s) + 1 + s f(s) - 1 - f'(s) = 0$$

or $s f(s) + (s^2+1)\,f'(s) = 0$

i.e., $\dfrac{f'(s)}{f(s)} = \dfrac{-s}{s^2+1} = -\dfrac{1}{2}\cdot\dfrac{2s}{s^2+1}.$

Integrating with regard to 's'

$\log f(s) = -\dfrac{1}{2}\log(s^2+1) + \log C$, C being constant of integration

$$\therefore\ f(s) = \frac{C}{\sqrt{s^2+1}}$$

Applying initial value theorem, we have

$$\lim_{s\to\infty} s f(s) = \lim_{t\to 0} F(t)$$

i.e., $\lim_{s\to\infty} \frac{sC}{\sqrt{s^2+1}} = \lim_{t\to 0} J_0(t)$ which gives C=1

Hence $f(s) = \frac{1}{\sqrt{s^2+1}}$ i.e., $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

Now using the change of scale of property, we have

$$L\{J_0(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right) \quad \text{where} \quad f(s) = L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

so that $f\left(\frac{s}{a}\right) = \frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{a}{\sqrt{s^2+a^2}}$

Hence $$L\{J_0(at)\} = \frac{a}{\sqrt{s^2+a^2}} \qquad \text{............ (26)}$$

Further to deduce $L\{J_0(t)\}$, using the property [13.3.6]

we get

$$L\{t J_0(at)\} = -\frac{d}{ds}\left[L\{J_0(at)\}\right] = -\frac{d}{ds}\left(\frac{a}{\sqrt{s^2+a^2}}\right) = \frac{as}{\left(s^2+a^2\right)^{3/2}} \qquad \text{.......... (27)}$$

Similarly $$L\{t J_1(t)\} = \frac{1}{(s^2+1)^{3/2}} \qquad \text{............... (28)}$$

13.4.3 The Error function and its complement

The error function of a variable t denoted by erf (t) or $E_r(t)$ is defined as

$$\text{erf}\,(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx = E_r(t)$$

and the complement of the error function denoted by erf C (t) is defined by

$$\text{erf C}(t) = 1 - \text{erf}(t) = 1 - \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.$$

It is notable that

$$\lim_{t\to 0} \text{erf}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \text{erf}(t) = 1.$$

Thus $\text{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} \left\{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\ldots\ldots\right\} dx$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\ 2!} - \frac{t^{7/2}}{7\ 3!} + \ldots\ldots\ldots\ldots\right]$$

$$\therefore \quad L\left\{\text{erf}\sqrt{t}\right\} = \frac{2}{\sqrt{\pi}}\, L\left\{t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\ 2!} - \frac{t^{7/2}}{7\ 3!} + \ldots\ldots\ldots\ldots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{s^{5/2}} + \frac{\Gamma 7/2}{s^{7/2}} - \frac{\Gamma 9/2}{s^{9/2}} + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\cdot\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\,\frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6}\,\frac{1}{s^{9/2}} + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}} \qquad \ldots\ldots\ldots\ldots\ (29)$$

13.4.4 The sine, cosine and exponential integrals

The sine integral is defined as $S_i(t) = \int_0^t \frac{\sin x}{x}\,dx$

and the cosine integral is defined as $C_i(t) = \int_0^t \frac{\cos x}{x}\,dx$

Also the exponential integral is defined as $E_i(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx$

we have $S_i(t) = \int_0^t \frac{\sin x}{x}\,dx = \int_0^t \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\right) dx$

$$= \int_0^t \left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\ldots\right\} dx$$

$$= t - \frac{t^3}{3\,3!} + \frac{t^5}{5\,5!} - \frac{t^7}{7\,7!} + ...$$

$$\therefore\ L\{S_i(t)\} = L\left\{t - \frac{t^3}{3\,3!} + \frac{t^5}{5\,5!} - \frac{t^7}{7\,7!} + ...\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3\ 3!}\cdot\frac{3!}{s^4} + \frac{1}{5\,5!}\cdot\frac{5!}{s^6} - \frac{1}{7\,7!}\cdot\frac{7!}{s^8} + ...$$

$$= \frac{1}{s}\left[\frac{1/s}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + ...\right]$$

$$= \frac{1}{s}\ \tan^{-1}\frac{1}{s} \qquad .... (30)$$

Similarly $L\{C_i(t)\} = \frac{1}{2s}\ \log\,(s^2+1)$ .... (31)

and $L\{E_i(t)\} = L\left\{\int_t^\infty \frac{e^{-x}}{x}\ dx\right\}$, Put $x = ty$ i.e., $\frac{dx}{x} = \frac{dy}{y}$

(on logarithmic differentiation)

$$= L\left\{\int_1^\infty \frac{e^{-ty}}{y}\ dy\right\}$$

$$= \int_0^\infty e^{-st}\left\{\int_1^\infty \frac{e^{-ty}}{y}\ dy\right\} dt$$ by definition of Laplace transform

$$= \int_1^\infty \frac{1}{y}\left\{\int_0^\infty e^{-(s+y)t}\ dt\right\}\ dy\ ,$$ by changing the order of integration

$$= \int_1^\infty \frac{1}{y}\cdot\frac{1}{s+y}\ dy$$

$$= \int_1^\infty \frac{1}{s}\left[\frac{1}{y} - \frac{1}{s+y}\right]\ dy\ = \ \frac{1}{s}\left[\log\,y - \log\,(s+y)\right]_1^\infty$$

$$= \frac{1}{s}\left[-\log\left(\frac{s}{y}+1\right)\right]_1^{\infty}$$

$$= \frac{1}{s}\log(s+1) \qquad \text{.............. (32)}$$

13.4.5 Heaviside Unit Function or Unit Step Function :

Unit step function is defined as

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

$$\therefore L\{U(t-a)\} = \int_0^{\infty} e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} . 0\, dt + \int_a^{\infty} e^{-st} . 1\, dt = \left.\frac{e^{-st}}{-s}\right|_a^{\infty}$$

$$= \frac{1}{s} e^{-as} \qquad \text{........... (33)}$$

13.4.6 Dirac delta function :

The Kronecker delta function is a function of two parameters, say m and n and we have the following inportant property

$$\sum_m C_m\, \delta_{mn} = C_n \qquad \text{........... (34)}$$

where C_m is a function of the discrete variable m. That is, δ_{mn} makes all the terms in (34) vanishing for $m \neq n$ and the only non zero term is $C_n\, \delta_{nn}$ which is equal to C_n. In fact, eqn. (34) itself may be taken as the difinition of Kronecker delta.

The Dirac delta function $\delta(x-x')$ plays the same role as a function of a continuous variable where as Kronecker delta is for discrete variables. Analogous to equ. (34), the Dirac delta function is defined as

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} f(x)\, \delta(x-x')\, dx = f(x') \\ \text{Also} \quad &\int_{-\infty}^{\infty} f(x)\, \delta x\, dx = f(0) \end{aligned}\right\} \qquad \text{...................... (35)}$$

which shows that the delta function in the integrand picks out the value of f(x) at the single point x' and does not take into account the behaviour of f(x) anywhere else.

It is also implied from (35) that

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} 1\ \delta(x-x')\,dx = 1 \\ \text{Also}\quad &\delta(x-x') = 0 \qquad (x \neq x') \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (36)$$

The delta function may be considered as a limit of certain ordinary functions such as given here.

(i) $\delta(x) = \underset{\in \to 0}{\text{Lt}}\ \delta_{\in}(x)$ where

$$\delta_{\in}(x) = \begin{cases} \dfrac{1}{2\in} & |x| < \in \\ 0 & |x| > \in \end{cases} \quad \ldots\ldots\ldots\ldots \quad (37)$$

(ii) $\delta(x) = \underset{\in \to 0}{\text{Lt}}\ \dfrac{1}{\sqrt{2\pi}\,\in}\, e^{-x^2/2\in^2}$ .... (38) Gaussian function.

Laplace transform of Dirac δ - function is

$$L\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)\, e^{-st}.\ dt = e^{-sa} \qquad \text{by (35)}$$

At a=0, $L\{\delta(t)\} = 1.$

Fourier transform of Dirac δ -function is

$$F\{\delta(t-a)\} = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-ist}.\ dt = e^{-isa} \qquad \text{from (35)}$$

and $F\{\delta(t)\} = 1.$

13.4.7 Obtain the Laplace transform of $\sin\sqrt{t}$:

Using series expansion method,

$$L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \ldots\ldots\ldots\ \ldots\ldots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} . L\{t^{5/2}\} - \frac{1}{7!} . L\{t^{7/2}\} + \ldots\ldots\ldots\ldots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\, s^{5/2}} + \frac{\Gamma(7/2)}{5!\, s^{7/2}} - \frac{\Gamma(9/2)}{7!\, s^{9/2}} + \ldots\ldots\ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \frac{1}{2^2 s} + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \frac{1}{3!}\left(\frac{1}{2^2 s}\right)^3 + \ldots\ldots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[e^{-\frac{1}{2^2 s}}\right] = \frac{e^{-s/4} . \sqrt{\pi}}{2\, s^{3/2}} \qquad \ldots\ldots\ldots\ldots \text{(39)}$$

13.4.8 Evaluation of Certain integrals by Laplace Transforms :

I Show that $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Sol :- Consider $G(t) = \int_0^\infty \cos tx^2\, dx$

Taking Laplace transforms on both sides

$$L\{G(t)\} = \int_0^\infty e^{-st}\left\{\int_0^\infty \cos tx^2\, dx\right\} dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-st} \cos tx^2\, dt\right) dx$$

$$= \int_0^\infty \frac{s}{s^2 + x^4}\, dx$$

Put $x^2 = s\tan\theta \qquad 2x\, dx = s \sec^2\theta\, . d\theta$

$$= \int_0^{\pi/2} \frac{s^2 \sec^2\theta\, d\theta}{2\sqrt{s\tan\theta}\, .\, s^2 \sec^2\theta}$$

or $dx = \frac{s \sec^2\theta}{2\sqrt{s\tan\theta}} . d\theta$

$$= \frac{1}{2\sqrt{s}} . \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} = \frac{1}{2\sqrt{s}} . \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{\frac{1}{2}}\theta\, . d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma\left(\frac{1}{4}\right) . \Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)}$$

using $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, . d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{\pi}{2 \sin\frac{\pi}{4}}$$ using $\Gamma(n)\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ $\quad 0<n<1$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{\pi\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4} \cdot \frac{1}{\sqrt{s}}.$$

$\therefore \quad G(t) = \frac{\sqrt{2\pi}}{4} L^{-1}\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{2\pi}}{4} \cdot \frac{1}{\sqrt{t}}$ by (23)

Putting $t = 1$,

$$G(1) = \int_0^\infty \cos x^2\ dx = \frac{\sqrt{2\pi}}{4} = \frac{1}{2} \cdot \sqrt{\frac{\pi}{2}}.$$

II. Show that $\int_0^\infty e^{-x^2}.\,dx = \frac{1}{2}\sqrt{\pi}$

Sol : Consider $G(t) = \int_0^\infty e^{-tx^2}.\,dx$ so that

$$L\{G(t)\} = \int_0^\infty e^{-st}\left(\int_0^\infty e^{-tx^2}.\ dx\right).\ dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-(s+x^2)t}.\ dt\right).\ dx$$

$$= \int_0^\infty \frac{dx}{s+x^2} = \frac{1}{\sqrt{s}} \tan^{-1}\frac{x}{\sqrt{s}}\Big|_0^\infty = \frac{\pi}{2\sqrt{s}}.$$

$$\therefore \quad G(t) = L^{-1}\left(\frac{\pi}{2\sqrt{s}}\right) = \frac{\sqrt{\pi}}{2}.L^{-1}\left(\frac{\sqrt{\pi}}{\sqrt{s}}\right) = \frac{\sqrt{\pi}}{2}.\,t^{-1/2}$$

or $\quad \int_0^\infty e^{-tx^2}\ dx = \frac{\sqrt{\pi}}{2}.\,t^{-1/2}$

Putting t=1, $\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}.$

III. Evaluate the integral $\int_0^\infty \frac{\cos tx}{x^2+a^2}.\,dx$.

Sol : Consider $F(t) = \int_0^\infty \frac{\cos tx}{x^2+a^2} . dx$

Now $L\{F(t)\} = \int_0^\infty e^{-st}\left(\int_0^\infty \frac{\cos tx}{x^2+a^2} . dx\right) dt$

$$= \int_0^\infty \frac{1}{x^2+a^2}\left(\int_0^\infty e^{-st} . \cos tx \ dt\right) dx$$

$$= \int_0^\infty \frac{1}{x^2+a^2} . \frac{s}{s^2+x^2} . dx$$

$$= \frac{s}{s^2-a^2} \int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right) dx$$

$$= \frac{s}{s^2-a^2} \left[\frac{1}{a} \tan^{-1}\frac{x}{a} - \frac{1}{s} \tan^{-1}\frac{x}{s}\right]_{x=0}^{\infty}$$

$$= \frac{\pi}{2a} \frac{1}{s+a} \qquad \text{for } a>0, \quad s>0$$

$$\therefore \quad F(t) = \frac{\pi}{2a} . L^{-1}\left\{\frac{1}{s+a}\right\} = \frac{\pi}{2a} e^{-at} \qquad (a>0, \ t \geq 0).$$

IV. Evaluate the integral

$$F(t) = \int_0^\infty \frac{\sin tx}{x} dx.$$

When $t > 0$ and $s > 0$, $L\{F(t)\} = \int_0^\infty e^{-st}\left[\int_0^\infty \frac{\sin tx}{x} dx\right] dt$

or $L\{F(t)\} = \int_0^\infty \frac{1}{x}\left(\int_0^\infty e^{-st} . \sin tx \ dt\right) dx$

$$= \int_0^\infty \frac{1}{x} \frac{x}{s^2+x^2} . dx = \int_0^\infty \frac{dx}{x^2+s^2} = \frac{1}{s} \tan^{-1}\frac{x}{s}\bigg|_0^\infty$$

$$= \frac{\pi}{2} . \frac{1}{s}$$

$$\therefore \quad F(t) = \frac{\pi}{2} \quad \text{and} \quad F(0) = 0 \qquad t \geq 0.$$

13.5 A Short table of Laplace transforms :

	F(t)	f(s)
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n \quad (n = 1,2,.......)$	$\frac{n!}{s^{n+1}}$
4.	$t^n.e^{at} \ (n = 1,2,.....)$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\sin kt$	$\frac{k}{s^2+k^2}$
6.	$\cos kt$	$\frac{s}{s^2+k^2}$
7.	$\sinh kt$	$\frac{k}{s^2-k^2}$
8.	$\cosh kt$	$\frac{s}{s^2-k^2}$
9.	$e^{-at} \sin kt$	$\frac{k}{(s+a)^2+k^2}$
10.	$e^{-at} \cos kt$	$\frac{s+a}{(s+a)^2+k^2}$
11.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2\sqrt{s^3}}$
12.	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
13.	$t^k \ (k > -1)$	$\frac{\Gamma(k+1)}{s^{k+1}}$

Additional Problems :

(1) Find the Laplace transform of $\frac{2}{t}\sinh t$.

Sol:- Let $F(t) = \frac{2}{t}\sinh t$

$$L\{F(t)\} = \int_0^\infty \frac{2}{t} . \sinh t . e^{-st} . dt$$

$$= \int_0^\infty \frac{2}{t} . \frac{e^t - e^{-t}}{2} . e^{-st} . dt$$

$$= \int_0^\infty \frac{1}{t} . e^{-(s-1)t} dt - \int_0^\infty \frac{1}{t} . e^{-(s+1)t} dt$$

$$= \int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \qquad \because L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du .$$

$$= \log \frac{u-1}{u+1}\Bigg|_s^\infty = \log 1 - \log\frac{s-1}{s+1} = \log\frac{s+1}{s-1} .$$

(2) What is the Laplace transform of $e^{kt}\cos(a+bt)$?

Sol:- Let $F(t) = e^{kt}\cos(a+bt)$

$$= e^{kt}\left(\cos a . \cos bt - \sin a . \sin bt\right)$$

$$= \cos a . e^{kt} . \cos bt - \sin a . e^{kt} . \sin bt$$

where a, b and k are constants.

$$\therefore L\{F(t)\} = \cos a . L\{e^{kt} . \cos bt\} - \sin a . L\{e^{kt} . \sin bt\}$$

$$= \cos a . \frac{s-k}{(s-k)^2 + b^2} - \sin a . \frac{b}{(s-k)^2 + b^2}$$

$$= \frac{(s-k)\cos a - b\sin a}{(s-k)^2 + b^2} .$$

(3) Find $L\{e^{-at}\sin^2(wt+\theta)\}$ where a, w, θ are constants.

Sol:- Let $F(t) = e^{-at}\sin^2(wt+\theta)$

$$= e^{-at}\left[\frac{1-\cos\left(2\,\overline{wt+\theta}\right)}{2}\right]$$

$$= \frac{1}{2}e^{-at} - \frac{\cos 2\theta}{2}\,.\,e^{-at}\cos 2wt + \frac{\sin 2\theta}{2}\,e^{-at}\sin 2wt$$

$$\therefore \quad L\{F(t)\} = \frac{1}{2(s+a)} - \frac{\cos 2\theta}{2}\cdot\frac{s+a}{(s+a)^2+4w^2} + \frac{\sin 2\theta}{2}\cdot\frac{2w}{(s+a)^2+4w^2}.$$

13.7 Summary of the Lesson

With a basic understanding of the linear integral transform, Laplace transforms have been defined. Several of the properties of Laplace transforms are proved and the examples are given for each property.

Laplace transforms involving Gamma function, Zeroth order Bessel function, Error function, etc., are specially treated. Dirac delta function and its Laplace transforms are clearly explained.

The use of Laplace transforms in the evaluation of certain integrals is explianed with specific examples.

A short table of transforms is given for easy memory and quick reference.

Some more examples have been worked and self assessment questions are given covering the entire lesson.

13.8 Key terminology

Integral transform - Laplace transform - Kernel function - Gamma function - Bessel function - Error function - Unit step function - Dirac delta function.

13.9 Self assessment questions

1. Using Laplace transform of derivatives, find $L\{t \sin wt\}$.

   Verify by applying the formula for the derivatives of Laplace transform.

2. Find the Laplace transform of $e^{kt}\, U_a(t)$ where

$$U_a(t) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

3. Find the Laplace transform of the saw-tooth wave function

$$F(t) = \frac{k}{p}\, t \qquad 0 < t < p \qquad \text{where} \qquad F(t+p) = F(t)$$

4. Find the Laplace transform of half-wave rectification of $-\sin wt$ as shown in figure.

5. Find the Laplace transform of

(i) $\frac{\sin kt}{t}$ (ii) $F(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t \end{cases}$

6. Obtain the Laplace transform of the differential equation

$$Y'(t) + 4Y(t) + 3\int_0^t Y(t)\ dt = F(t)$$

where $F(t) = \begin{cases} 1 & 0 < t < 2 \\ -1 & 2 < t < 4 \end{cases}$ is a periodic function.

13.10 Reference Books

1. B.D. Gupta. “Mathematical Physics” Vikas Publishing House, 1980.
2. R.V. Churchill “Operational Mathematics” McGraw-Hill Book Co., 1958.
3. E. Kreyszig “Advanced Engineering Mathematics” Wiley Eastern Pvt. Ltd., 1971.

Unit - IV

Lesson - 14

INVERSE LAPLACE TRANSFORMS

Objective of the lesson :

* To define the inverse Laplace transform
* To define convolution and to prove a theorem
* To state and prove Heaveside expansion theorem
* To give number of worked examples.

Structure of the lesson :

14.1. Introduction :

Much about Laplace transforms has been exposed in the previous lesson but using inverse Laplace transforms finally will help us to complete the solution of the given problem. Thus the usage of Laplace transform and its invrse is an intermediate technique for a convenient and easy approach to solve the problem. Out of all the integral transforms within our reach, the adaptability of Laplace transform and its inverse is supposed to be simple in view of their form, existence and uniqueness.

14.2 Inverse Laplace Transforms :

Let the symbol $L^{-1}\left\{f(s)\right\}$ denote a function $F(t)$ whose Laplace transform is $f(s)$.

Thus if $L\{F(t)\} = f(s)$

then $F(t) = L^{-1}\{f(s)\}$

For instance, as seen from the table of Laplace transforms in Lesson 13,

$$L^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt} = F_1(t) \text{ (say) ---------- (1)}$$

This correspondence between $F_1(t)$ and the function of 's' shows that $F_1(t)=e^{kt}$ is an inverse transform of $\frac{1}{s-k}$

But another function

$$\left.\begin{array}{l} F_2(t)=e^{kt} \\ \quad\quad\;\; =1 \end{array}\right\} \begin{array}{c} 0<t<2 \text{ and } t>2 \\ \text{for } t=2 \end{array} \quad \text{------------ (2)}$$

has the transform as

$$L\{F_2(t)\} = \int_0^\infty e^{-st} F_2(t)\,dt = \int_0^2 e^{-st} . e^{kt}\,dt + \int_2^\infty e^{-st} e^{kt}\,dt$$

and this is the same as $L\{F_1(t)\}$ in (1).

The function $F_2(t)$ could be chosen equally well as one that differs from $F_1(t)$ at any finite set of values of t or even at such an infinite set as t = 1, 2, 3, ... So we have seen that for two different functions $F_1(t)$ and $F_2(t)$, the transform is the same.

However, a theorem on the uniqeness of the inverse transform, due to Lerch, states that if two functions $F_1(t)$ and $F_2(t)$ have the same Laplace transform $f(s)$, then

$$F_2(t)=F_1(t)+N(t)$$

Here $N(t)$ is a null function such that $\int_0^T N(t)\,dt = 0$ for every +ve 'T'.

In view of this theorem, we can say that the inverse transform is essentially unique since a null function is usually of no importance in the applications.

In particular, if two continuous functions have the same transform, they are completely identical.

Thus all the formulae or properties derived in lesson 13 hold good for inverse transforms also.

For example, we may state the first translation property as

"The substitution of $s-a$ for the variable s in the transform $f(s)$ corresponds to the multiplication of the object function $F(t)$ by the function e^{at}".

Proof : We have

$$f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)dt = \int_0^\infty e^{-st}\left[e^{at} F(t)\right]dt$$

$$= L\left\{e^{at} F(t)\right\}$$

Similarly all the other properties can be thought of in terms of inverse transform.

14.3. Convolution :

The <u>Convolution</u> F^*G of the function $F(t)$ and $G(t)$ is defined as the function

$$F(t)^* G(t) = \int_0^t F(\lambda)G(t-\lambda)d\lambda \text{ ----------- (3)}$$

(This is also called <u>Convolution</u> or <u>Faltung</u> integral)

The Convolution operation is commutative :

$$F(t)^* G(t) = \int_0^t F(\lambda)G(t-\lambda)d\lambda$$

$$= -\int_t^0 G(u)F(t-u)du \qquad \text{Put } t-\lambda = u\text{ , } \therefore d\lambda = -du$$

$$= \int_0^t G(u)\, F(t-u)du$$

$$= G(t)^* F(t)$$

Some more properties are

$$F(t)^*\left[G(t)+H(t)\right] = F(t)^* G(t)+F(t)^* H(t)$$

$$F(t)^* kG(t) = k\left[F(t)^* G(t)\right]$$

$$F(t)^*\left[G(t)^* H(t)\right]=\left[F(t)^* G(t)\right]^* H(t)$$

Theorem - 1 : Prove that

$$L\left\{F(t)^* G(t)\right\} = L\left\{F(t)\right\} L\left\{G(t)\right\}$$

$$= f(s)\ g(s)$$

Proof : We have, by definition

$$L\left\{F(t)^* G(t)\right\} = L\left\{\int_0^t F(t-\lambda)G(\lambda)d\lambda\right\} \text{ --------- (4)}$$

Consider the unit step function

$$U(t-\lambda)=\begin{cases}1 & \lambda<t\\ 0 & \lambda>t\end{cases} \quad \text{with } \lambda \text{ as variable ------(5)}$$

or

$$=\begin{cases}1 & t>\lambda\\ 0 & t<\lambda\end{cases} \quad \text{with '}t\text{' as variable ------(6)}$$

Using (5), Equation (4) can be written as

$$L\left\{\int_0^t F(t-\lambda)G(\lambda)d\lambda\right\} = \int_0^\infty\left[\int_0^t F(t-\lambda)G(\lambda)d\lambda\right]e^{-st}\ dt$$

$$=\int_0^\infty\left[\int_0^\infty F(t-\lambda)G(\lambda)\cup(t-\lambda)d\lambda\right]e^{-st}\ dt$$

$$=\int_0^\infty G(\lambda)\left[\int_0^\infty e^{-st} F(t-\lambda)\cup(t-\lambda)dt\right]d\lambda \text{ -------- (7)}$$

(by interchanging the order of integration)

Because of the presence of $U(t-\lambda)$, the integrand of the inner integral with respect to t is identically zero for all $t<\lambda$ from (6). Hence, the inner integration effectively starts not at $t=0$, but at $t=\lambda$. Therefore, (7) becomes

$$L\left\{\int_0^t F(t-\lambda)G(\lambda)\,d\lambda\right\}=\int_0^\infty G(\lambda)\left[\int_\lambda^\infty F(t-\lambda)e^{-st}\,dt\right]d\lambda \text{ --------- (8)}$$

Now in the inner integral on the right of (8), let $t-\lambda=\tau$ and $dt=d\tau$.

Then $$L\left\{\int_0^t f(t-\lambda)G(\lambda)\,d\lambda\right\}=\int_0^\infty G(\lambda)\left[\int_0^\infty F(\tau)e^{-s(\tau+\lambda)}\,d\tau\right]d\lambda$$

$$=\int_0^\infty G(\lambda)e^{-s\lambda}\left[\int_0^\infty F(\tau)e^{-s\tau}\,d\tau\right]d\lambda$$

$$=\left[\int_0^\infty F(\tau)e^{-s\tau}\,d\tau\right]\left[\int_0^\infty G(\lambda)e^{-s\lambda}\,d\lambda\right]$$

$$=L\{F(t)\}\;L\{G(t)\}$$

$$=f(s)\;g(s)$$

So $$L\left\{F(t)^*G(t)\right\}=L\left\{G(t)^*F(t)\right\}$$

$$=f(s)g(s) \text{ ----------------- (9)}$$

Note : *The advantage of this theorem is seen to be appropriate while finding the inverse Laplace transform of a product of functions of s for which the inverse Laplace transforms of individual factors are known. This can be seen in forth coming examples.*

14.4 Partial Fraction Methods :

For a given function of s, wherever it is possible, one can use the usual methods of putting that function into partial fractions so that the inverse Laplace transforms can be found using the formulae or the table of Laplace transforms.

However, the following Heaveside expansion theorems are of great utility in this connection.

Theorem - 2 : If $F(t)=L^{-1}\left[p(s)/q(s)\right]$ where $p(s)$ and $q(s)$ are polynomials and the degree of $q(s)$ is greater than the degree of $p(s)$, then the term in $F(t)$ corresponding to an unrepeated linear factor $(s-a)$ of $q(s)$ is

$$\frac{p(a)}{q'(a)}e^{at} \quad \text{or equally well} \quad \frac{p(a)}{h(a)}e^{at} \quad \text{------------ (10)}$$

where $h(s)$ is the product of all the factors of $q(s)$ except $(s-a)$.

Proof : From the familiar partial fraction decomposition of $\frac{p(s)}{q(s)}$, we can write

$$\frac{p(s)}{q(s)}=\frac{A}{s-a}+\phi(s) \quad \text{------------ (11)}$$

Since $(s-a)$ is an unrepeated factor of $q(s)$, $\phi(s)$ remains finite as s approaches a. Multiplying (11) by $(s-a)$ and taking the limit as $s\to a$, we get

$$\underset{s\to a}{Lt}\frac{(s-a)\,p(s)}{q(s)}=\underset{s\to a}{Lt}\frac{p(s)}{q(s)/(s-a)}=\underset{s\to a}{Lt}\left[A+(s-a)\phi(s)\right]$$

or $$\underset{s\to a}{Lt}\frac{p(s)}{q(s)/(s-a)}=A+0 \quad \text{-------- (12)}$$

The limit of the numerator in (12) is obviously $p(a)$. The denominator is an indeterminate quantity at $s=a$ and hence applying L' Hospital's rule.

We get $\underset{s\to a}{Lt}\frac{q(s)}{s-a}=\frac{q'(a)}{1}=q'(a)$ or $h(a)$ if $(s-a)$ factor is cancelled in $\frac{q(s)}{s-a}$

So $A=\frac{p(a)}{q'(a)}$ or $\frac{p(a)}{h(a)}$

So, while taking the inverse Laplace transform of (11), it is clear that the fraction term $\frac{A}{s-a}$ gives rise to the term $Ae^{at} = \frac{p(a)}{q'(a)}e^{at} = \frac{p(a)}{h(a)}e^{at}$ as asserted.

Theorem - 3 : If $F(t)=L^{-1}\left[\frac{p(s)}{q(s)}\right]$, where $p(s)$ and $q(s)$ are polynomials and the degree of $q(s)$ is greater than the degree of $p(s)$, then the terms in $F(t)$ corresponding to a repeated linear factor $(s-a)^2$ in $q(s)$ are

$$\left[\frac{h^{(r-1)}(a)}{\lfloor r-1}+\frac{h^{(r-2)}(a)}{\lfloor r-2}.\frac{t}{\lfloor 1}+.......+\frac{h'(a)}{\lfloor 1}.\frac{t^{r-2}}{\lfloor r-2}+h(a)\frac{t^{r-1}}{\lfloor r-1}\right]e^{at} \quad \text{--------- (13)}$$

where $h(s)$ is the quotient of $p(s)$ and all the factors of $q(s)$ except $(s-a)^r$.

Proof : From the theory of partial fraction, a repeated linear factor $(s-a)^r$ of $q(s)$ gives rise to the component fractions.

$$\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+......+\frac{A_{r-1}}{(s-a)^{r-1}}+\frac{A_r}{(s-a)^r}$$

Let $\phi(s)$ be equal to the sum of the fractions corresponding to all the other factors of $q(s)$. Then, we have

$$\frac{p(s)}{q(s)}=\frac{h(s)}{(s-a)^r}=\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+......+\frac{A_{r-1}}{(s-a)^{r-1}}+\frac{A_r}{(s-a)^r}+\phi(s)$$

or $$h(s)=A_1(s-a)^{r-1}+A_2(s-a)^{r-2}+......A_{r-1}(s-a)+A_r+(s-a)^r\phi(s)$$

$$\therefore h(a)=A_r$$

Again,

$$h'(s)=A_1(r-1)(s-a)^{r-2}+A_2(r-2)(s-a)^{r-3}+.....+A_{r-1}+(s-a)^r\phi'(s)+r(s-a)^{r-1}.\phi(s)$$

setting $s=a$, we get

$$h'(a)=A_{r-1}$$

Continuing in this fashion, we obtain

$$h''(a)=\lfloor 2\, A_{r-2}$$

$$h'''(a)=\lfloor 3\, A_{r-3}$$

- - - - - - - - - - - - -

- - - - - - - - - - - - -

$$h^{(r-1)}(a)=\lfloor r-1\; A_1$$

or $$A_{r-k}=\frac{h^{(k)}(a)}{\lfloor k} \qquad k=0,1,.....r-1$$

The terms in the expansion of $\frac{p(s)}{q(s)}$, which correspond to the factor $(s-a)^r$ one, therefore.

$$\frac{h^{(r-1)}(a)}{\lfloor r-1}\cdot\frac{1}{s-a}+\frac{h^{(r-2)}(a)}{\lfloor r-2}\frac{1}{(s-a)^2}+......+\frac{h'(a)}{\lfloor 1}\frac{1}{(s-a)^{r-1}}+h(a)\frac{1}{(s-a)^r}$$

Recalling that $L^{-1}\left\{\frac{1}{(s-a)^n}\right\}=\frac{t^{n-1}}{\lfloor n-1}e^{at}$ it is evident that the terms in $F(t)$, which arise from these fractions are

$$\left[\frac{h^{(r-1)}(a)}{\lfloor r-1}\frac{1}{s-a}+\frac{h^{(r-2)}(a)}{\lfloor r-2}\frac{t}{\lfloor 1}+........+\frac{h'(a)}{\lfloor 1}.\frac{t^{r-2}}{\lfloor r-2}+h(a)\frac{t^{r-1}}{\lfloor r-1}\right]e^{at}$$

as asserted in the Theorem.

Theorem - 4 : If $F(t)=L^{-1}\left[\frac{p(s)}{q(s)}\right]$, where $p(s)$ and $q(s)$ are polynomials and the degree of $q(s)$ is greater than the degree of $p(s)$, then the terms in $F(t)$ which correspond to an unrepeated, irreducible quadratic factor $(s+a)^2+b^2$ of $q(s)$ are

$$\frac{e^{-at}}{b}\left(h_i \cos bt + h_r \sin bt\right) \quad \text{----------- (14)}$$

Where h_r and h_i are respectively, the real and imaginary parts of $h(-a+ib)$ and $h(s)$ is the quotient of $p(s)$ and all the factors of $q(s)$ except the factor $(s+a)^2+b^2$.

Proof : An unrepeated, irreducible quadratic factor $(s+a)^2+b^2$ of $q(s)$ gives rise to a single fraction of the form $\frac{As+B}{(s+a)^2+b^2}$ in the partial fraction of expansion of $\frac{p(s)}{q(s)}$. If $\phi(s)$ denotes the fractions corresponding to all the other factors of $q(s)$, we can write

$$\frac{p(s)}{q(s)}=\frac{h(s)}{(s+a)^2+b^2}=\frac{As+B}{(s+a)^2+b^2}+\phi(s)$$

or $$h(s)=As+B+\left[(s+a)^2+b^2\right]\phi(s)$$

substituting $s=-a+ib$ which makes $(s+a)^2+b^2$ vanish, simplifies the above identity as

$$h(-a+ib)=(-a+ib)A+B$$

or $$(h_r+i\,h_i)=(-aA+B)+ibA$$

$$\therefore h_r=-aA+B \text{ and } h_i=bA$$

or $$A=\frac{h_i}{b}; \qquad B=\frac{bh_r+ah_i}{b}$$

Thus the partial fraction corresponding to the quadratic factor $(s+a)^2+b^2$ is

$$\frac{As+B}{(s+a)^2+b^2}=\frac{1}{b}\,\frac{h_i s+(b\,h_r+a\,h_i)}{(s+a)^2+b^2}$$

$$=\frac{1}{b}\left[\frac{(s+a)\,h_i}{(s+a)^2+b^2}+\frac{b\,h_r}{(s+a)^2+b^2}\right]$$

The inverse of this expression is evidently

$$\frac{1}{b}\left(h_i\cos bt+h_r\sin bt\right)e^{-at}$$

Note : *There is another theorem on repeated quadratic factors. But, due to its complexity and limited usefulness, it is not dealt with. When the need arises to deal with repeated quadratic factors, convolution theorem can be applied.*

14.5 Examples :

1. Using Laplace transform techniques. prove $\beta(m,n)=\frac{\lceil m\,\lceil n}{\lceil m+n}$

Solution : Let $G(t)=\int_0^t x^{m-1}(t-x)^{n-1}\,dx$

which gives beta function at t = 1

Now $L\{G(t)\}=L\{t^{m-1}\}\,L\{t^{n-1}\}$ applying convolution Theorem

$$=\frac{\lceil m}{s^m}\,\frac{\lceil n}{s^n}=\frac{\lceil m\,\lceil n}{s^{m+n}}$$

or $\qquad G(t)=L^{-1}\frac{\lceil m\,\lceil n}{s^{m+n}}=\lceil m\,\lceil n\,\frac{t^{m+n-1}}{\lceil m+n}$

since $\quad L^{-1}\frac{\lceil n+1}{s^{n+1}}=t^n$ (Equation (22) of lesson 13)

$$\therefore \int_0^t x^{m-1}(t-x)^{n-1}\,dx = \frac{\lceil m \lceil n}{\lceil m+n}\, t^{m+n-1}$$

Putting $t=1$, we get

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\lceil m \; \lceil n}{\lceil m+n}$$

2. Solve the differential equation $Y''(t)+2Y'(t)+Y(t)=t\,e^{-t}$ for which $Y(0)=1, Y'(0)=-2$

Solution : Taking the Laplace transforms on both sides of the differential equation, we get

$$\left[s^2 y(s)-sY(0)-Y'(0)\right]+2\left[sy(s)-Y(0)\right]+y(s)=\frac{1}{(s+1)^2}$$

using the given intital conditions,

$$s^2 y(s)-s+2+2sy(s)-2+y(s)=\frac{1}{(s+1)^2}$$

$$(s+1)^2 y(s)=\frac{1}{(s+1)^2}+s$$

or $$y(s)=\frac{1}{(s+1)^4}+\frac{s}{(s+1)^2}$$

$$=\frac{1}{(s+1)^4}+\frac{1}{(s+1)}-\frac{1}{(s+1)^2} \qquad (\because s=(s+1)-1)$$

Taking inverse Laplace transforms, we get

$$Y(t)=L\left\{\frac{1}{(s+1)^4}\right\}+L\left\{\frac{1}{s+1}\right\}-L\left\{\frac{1}{(s+1)^2}\right\}$$

$$=\frac{t^3 e^{-t}}{\lfloor 3}+e^{-t}-t\,e^{-t}$$ using first shifting property.

3. What is $F(t)$ if $L\{F(t)\} = \ln\frac{s+1}{s-1}$.

Solution : Given that $f(s)=\ln(s+1)-\ln(s-1)$

$$\therefore \frac{df(s)}{ds} = \frac{1}{s+1}-\frac{1}{s-1}$$

$$= L\{-t\ F(t)\} \text{ (from (14) of lesson 13)}$$

or $$-t\ F(t)=L^{-1}\frac{1}{s+1}-L^{-1}\frac{1}{s-1}$$

$$=e^{-t}-e^{t}$$

$$\therefore F(t)=\frac{1}{t}\left(e^{t}-e^{-t}\right) = \frac{2}{t}\sin ht$$

4. Find $F(t)$ if $f(s)=\frac{s}{\left(s^2-1\right)^2}$

Solution : From equation (16) of lesson 13, we have seen that $L^{-1}\int_s^\infty f(u)\,du = \frac{F(t)}{t}$

Now $$\int_s^\infty \frac{u\,du}{\left(u^2-1\right)^2}$$

$$= \int_{s^2-1}^\infty \frac{d\mathrm{v}}{2\mathrm{v}^2} \qquad \text{Put } u^2-1=\mathrm{v}; \qquad 2u\,du = d\mathrm{v}$$

$$=-\frac{1}{2}\left.\frac{1}{\mathrm{v}}\right|_{s^2-1}^{\infty} = \frac{1}{2\left(s^2-1\right)}$$

$$\therefore L^{-1}\frac{1}{2\left(s^2-1\right)}=\frac{F(t)}{t}$$

$$L^{-1}\frac{1}{2}\ \frac{1}{2}\left[\frac{1}{s-1}-\frac{1}{s+1}\right]=\frac{F(t)}{t}$$

or $$F(t)=\frac{t}{4}\left(e^{t}-e^{-t}\right)=\frac{t}{2}\sin ht$$

5. If $f(s)=\dfrac{s}{(s+2)^2\left(s^2+2s+10\right)}$ what is $F(t)$?

Solution : We know the Heaveside expansion theorem as one for a repeated linear factor and another for a unrepeated quadratic factor as follows.

i) If $f(s)=\dfrac{p(s)}{q(s)}=\dfrac{h(s)}{(s-a)^2}$ where $(s-a)^2$ is a repeated linear factor, then the terms in $F(t)$ corresponding to a linear factor are

$$\left[\frac{h'(a)}{\lfloor 1}+\frac{h(a)}{1}t\right]e^{at} \text{ --------- (13)}$$

ii) If $f(s)=\dfrac{p(s)}{q(s)}=\dfrac{h(s)}{(s+a)^2+b^2}$ where $(s+a)^2+b^2$ is an unrepeated quadratic factor, then the terms in $F(t)$ corresponding to that factor are

$$\frac{e^{-at}}{b}\left[h_i\cos bt+h_r\sin bt\right] \text{ -------- (14)}$$

where h_r and h_i respectively the real and imaginary parts of $h(-a+ib)$

In the present problem, in case (i) $h(s)=\dfrac{s}{s^2+2s+10.}$ Then the terms in $F(t)$ corresponding to the repeated linear factor $(s+2)^2$ are

$$\left[\frac{h'(-2)}{\lfloor 1}+h(a)t\right]e^{at}$$

Now $h'(s)=\dfrac{(s^2+2s+10)1-s(2s+2)}{(s^2+2s+10)^2}=\dfrac{-s^2+10}{(s^2+2s+10)^2}$

$$\therefore h'(-2)=\frac{6}{100};h(-2)=-\frac{2}{10}$$

$\therefore$ (13) becomes $\left(-\dfrac{1}{5}t+\dfrac{3}{50}\right)e^{-2t}$

Similarly in case (ii)

$$h(s)=\frac{s}{(s+2)^2}; \qquad a=1,\ b=3$$

$$\therefore h(-a+ib)=\frac{-1+3i}{(-1+3i+2)^2}=\frac{-1+3i}{(3i+1)^2}$$

$$=\frac{-1+3i}{6i-8}=\frac{(-1+3i)(3i+4)}{2(-9-16)}=\frac{-13+9i}{-50}$$

$$\therefore h_r=\frac{13}{50},\ h_i=-\frac{9}{50}$$

So (14) becomes

$$\frac{e^{-t}}{3}\left[-\frac{9}{50}\cos 3t+\frac{13}{50}\sin 3t\right]$$

$$F(t)=\frac{(3-10t)e^{-2t}}{50}+e^{-t}\,\frac{(-9\cos 3t+13\sin 3t)}{150}$$

Note : *One can throw the given function* $f(s)$ *into partial fractions by direct method and get the inverse Laplace transform.*

6. The electric current $I(t)$ and the charge $Q(t)$ on the capacitor in a LCR circuit satisfy the conditions

$$L\frac{dI}{dt}+RI+\frac{Q}{C}=E_0$$

$$Q=\int_0^t I(\lambda)\,d\lambda$$

$$Q_0=I(0)=0;\ E_0= \text{ constant}$$

(a) Derive the formula

$$I=\frac{E_0}{\omega_1 L}\,e^{-bt}\,\sin\omega_1 t$$

where $b=\dfrac{R}{2L}$ and $\omega_1^2=\dfrac{1}{Lc}-\dfrac{R^2}{4L^2}>0$

(b) If $k^2=\dfrac{R^2}{4L^2}-\dfrac{1}{Lc}>0$, show that

$$I=\frac{E_0}{kL}e^{-bt}\,\sin h\ kt$$

Solution : The given differential equation is

$$L\,I'(t)+RI(t)+\frac{Q(t)}{c}=E_0$$

whre $Q(t)=\int_0^t I(\lambda)\,d\lambda=I(t)^{*}1$

Taking Laplace transforms on both sides, we get the subsidiary equation as

$$L\left[s\,i(s)-I(o)\right]+R\,i(s)+\frac{i(s)}{cs}=\frac{E_0}{s}$$

or $$i(s)\left(Ls+R+\frac{1}{cs}\right)=\frac{E_0}{s}$$

$$\therefore i(s)=\frac{CE_0}{Lc\,s^2+Rcs+1}=\frac{CE_0}{LC}\ \frac{1}{s^2+\frac{R}{2}s+\frac{1}{LC}}$$

$$=\frac{E_0}{L}\ \frac{1}{\left(s+\frac{R}{2L}\right)^2+\left(\frac{1}{Lc}-\frac{R^2}{4L^2}\right)}$$

$$=\frac{E_0}{L}\ \frac{\sqrt{\frac{1}{Lc}-\frac{R^2}{4L^2}}}{\left(s+\frac{R}{2L}\right)^2+\left(\frac{1}{Lc}-\frac{R^2}{4L^2}\right)}\cdot\frac{1}{\sqrt{\frac{1}{Lc}-\frac{R^2}{4L^2}}}\quad \text{-------- (i)}$$

$$\therefore I(t)=\frac{E_0}{L\sqrt{\frac{1}{Lc}-\frac{R^2}{4L^2}}}\ \sin\left[\left(\frac{1}{Lc}-\frac{R^2}{4L^2}\right)t\right]e^{-\frac{R}{2L}t}\qquad \text{if } \frac{1}{Lc}-\frac{R^2}{4L^2}>0$$

as asserted in part (a) of the question.

When $\frac{R^2}{4L^2}-\frac{1}{Lc}>0,$ then (i) takes the form

$$i(s)=\frac{E_0}{L}\cdot\frac{\sqrt{\frac{R^2}{4L^2}-\frac{1}{Lc}}}{\left(s+\frac{R}{2L}\right)^2-\left(\frac{R^2}{4L^2}-\frac{1}{Lc}\right)}\cdot\frac{1}{\sqrt{\frac{R^2}{4L^2}-\frac{1}{Lc}}}$$

$$\therefore I(t) = \frac{E_0}{L\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{Lc}}\right)} \cdot \sinh\left[\left(\frac{R^2}{4L^2} - \frac{1}{Lc}\right)t\right] e^{\frac{-R}{2L}t}$$

as required in (b) of the question

7. If $f(s) = \frac{1}{\left(s^2 + 4s + 13\right)^2}$, find $F(t)$.

Solution : We can write

$$f(s) = \frac{1}{\left[(s+2)^2 + 3^2\right]^2} = \frac{1}{(s+2)^2 + 3^2} \cdot \frac{1}{(s+2)^2 + 3^2}$$

By first shifting property, we can write

$$L^{-1} f(s) = F(t) = e^{-2t}\ L^{-1} \frac{1}{\left(s^2 + 3^2\right)^2}$$

$$= e^{-2t}\ L^{-1}\left[\left(\frac{1}{s^2 + 3^2}\right)\frac{1}{\left(s^2 + 3^2\right)}\right]$$

$$= e^{-2t}\left[L^{-1}\left(\frac{1}{s^2 + 3^2}\right) * L^{-1}\left(\frac{1}{s^2 + 3^2}\right)\right]$$ by convolution theorem.

$$= \frac{e^{-2t}}{9}\left[\sin 3t * \sin 3t\right]$$

$$= \frac{e^{-2t}}{9} \int_0^t \sin 3\lambda\ \sin 3(t - \lambda)\, d\lambda$$

$$= \frac{e^{-2t}}{9} \int_0^t \frac{\sin(6\lambda - 3t) - \cos 3t}{2}\, d\lambda$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin(6\lambda - 3t)}{6} - \lambda\cos 3t\right]_{\lambda=0}^{t}$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

8. Solve the problem

$$Y'''(t) - 2Y''(t) + 5Y'(t) = 0$$

$$Y(0) = 0, Y'(0) = 1, Y\left(\pi/8\right) = 1$$

Solution : Taking Laplace transforms on both sides of the differential equation, we get

$$s^2 y(s) - s - k - 2s^2 y(s) + 2 + 5s\, y(s) = 0 \text{ where } Y''(0) = k$$

or $$y(s) = \frac{k-2+s}{s(s^2-2s+5)} = (k-2)\frac{1}{s}\frac{1}{(s-1)^2+2^2} + \frac{1}{(s-1)^2+2^2}$$

$$\therefore Y(t) = (k-2)L^{-1}\frac{1}{s}\frac{1}{(s-1)^2+2^2} + L^{-1}\frac{1}{(s-1)^2+2^2}$$

But $$L^{-1}\frac{1}{(s-1)^2+2^2} = \frac{e^{+t}}{2}L^{-1}\frac{2}{s^2+2^2} = \frac{e^{+t}}{2}\sin 2t$$

$$\therefore Y(t) = (k-2)\left[1 * \frac{e^t}{2}\sin 2t\right] + \frac{e^{+t}}{2}\sin 2t$$

$$= \frac{(k-2)}{2}\int_0^t e^{+\lambda}\sin 2\lambda\, d\lambda + \frac{e^{+t}}{2}\sin 2t$$ by Convolution Theorem

But $$\int_0^t e^{+\lambda}\sin 2\lambda\, d\lambda = \mathrm{I_m}\int_0^t e^{+\lambda} e^{i2\lambda}\, d\lambda$$

$$= \mathrm{I_m}\left[\frac{e^{(+1+2i)\lambda}}{+1+2i}\right]_0^{t_0}$$

$$= \mathrm{I_m}\,\frac{1}{+1+2i}\left[e^{(+1+2i)t}-1\right] = \mathrm{Im}\,\frac{+1-2i}{5}\left[e^{+t}\cos 2t + e^{+t}\, i\sin 2t - 1\right]$$

$$= \mathrm{I_m}\cdot\frac{(1-2i)}{5}\left[\left(e^{+t}\cos 2t - 1\right) + i\,e^{+t}\sin 2t\right]$$

$$= \frac{e^{+t}}{5}\sin 2t - \frac{2}{5}\left(e^{+t}\cos 2t - 1\right)$$

So, $Y(t) = \dfrac{k-2}{2}\,\dfrac{e^t}{5}\,\sin 2t - \dfrac{2}{5}\,\dfrac{(k-2)}{2}e^t\cos 2t + \dfrac{2}{5}\,\dfrac{k-2}{2} + \dfrac{e^t}{2}\sin 2t$

$$= \frac{k-2}{5} + e^t\left[\frac{k+3}{10}\sin 2t - \frac{k-2}{5}\cos 2t\right]$$

since $Y\left(\pi/8\right) = 1,$ it follows that

$$1 = \frac{k-2}{5} + \frac{e^{\pi/8}}{10\sqrt{2}}\left(k+3-2k+4\right)$$

or $\dfrac{k-7}{5} + \dfrac{e^{\pi/8}}{10\sqrt{2}}(7-k) = 0$ or $k = 7$

Hence the solution is

$$Y(t) = 1 + e^t\left(\sin 2t - \cos 2t\right).$$

Note : *One can also proceed with the problem by putting into partial fractions by direct method.*

9. Find the general solution of the differential equation $Y''(t) + k^2 Y(t) = F(t)$ in terms of the constatnt k and $F(t)$.

Solution : By taking the Laplace transforms on both sides, we get

$$s^2 y(s) - sY(0) - Y'(0) + k^2 y(s) = f(s)$$

where $Y(0)$ and $Y'(0)$ are arbitrary constants.

so $$y(s) = \frac{1}{k}\frac{k}{s^2+k^2} f(s) + Y(0)\frac{s}{s^2+k^2} + \frac{Y'(0)}{k}\frac{k}{s^2+k^2}$$

$$\therefore Y(t) = \frac{1}{k}\sin kt^{*} F(t) + Y(0)\cos kt + \frac{Y'(0)}{k}\sin kt$$

$$\therefore Y(t) = \frac{1}{k}\int_0^t \sin k\lambda \; F(t-\lambda)\, d\lambda + C_1 \cos kt + C_2 \sin kt$$

whre C_1 and C_2 are arbitrary constants is the general solution :

10. Solve the integral equation $Y(t) = at + \int_0^t Y(\lambda)\sin(t-\lambda)\, d\lambda$

Solution : We can write the equation as $Y(t) = at + Y(t)^{*} \sin t$

Then, by taking Laplace transforms on both sides, we get

$$y(s) = \frac{a}{s^2} + \frac{y(s)}{s^2+1}$$

or $$y(s) = \frac{a}{s^2}\frac{s^2+1}{s^2} = a\left(\frac{1}{s^2} + \frac{1}{s^4}\right)$$

$$\therefore Y(t) = a\left(t + \frac{1}{6}t^3\right)$$

which is the solution of the given integral equation.

14.6 Summary of the Lesson :

In every application of Laplace transforms, both Laplace transforms and their inverses will be used. So this lesson is continuous to the previous lesson. Convolution theorem is the highlight of this lesson. Three Heaveside expansion theorems are given. These will be helpful when the direct method of putting the given $f(s)$ into partial fractions involves complexities. Different types of problems have been worked.

14.7 Key Terminology :

Lerch theorem - Convolution - Heavside expansion - integral equation.

14.8 Self-assessment Questions :

1. Find the inverse Laplace transforms of

(i) $\frac{1}{s}\tan^{-1}\frac{1}{s}$ (ii) $\frac{s+2}{(s+1)(s^2+4)}$ (iii) $\frac{s}{(s^2+a^2)(s^2+b^2)}$

2. Solve the differential equation

$Y^{(4)}(t)+Y'''(t)=\cos t$ given that $Y(0)=Y'(0)=Y'''(0)$ and $Y''(0)=k$

3. Solve the integral equation

$$F(t)+2\int_0^t F(x)\cos(t-x)\,dx = 9e^{2t}$$

4. Obtain the solution of

$X''(t)+4X'(t)+4X(t)=4e^{-2t}$ so that $X(0)=-1,\ X'(0)=4$

5. Find the solution of

$Y''(t)-k^2Y(t)=F(t)$ interms of k and $F(t)$ where $Y(0)=Y'(0)=0$, $k\neq 0$.

14.9 Reference Books :

1. C.R. Wylie Jr. : 'Advanced Engineering Mathematics' - Mc-GrawHill
2. R.V. Churchill : 'Operational Mathematics' Mc-Graw Hill, 1958
3. B.S. Rajput : 'Mathematical Physics' Pragati Prakashan 1999.

Unit - IV

Lesson - 15

FOURIER SERIES

Objective of the lesson :

* To define Fourier Series for a function having a period of 2π
* To derive Fourier series for even and odd functions and for functions of arbitrary period.
* To explain and treat half-range Fourier expansions.
* To bring out alternative forms of Fourier Series.
* To derive Fourier Integral as a limit of the Fourier Series.
* To define generalized Fourier Series.

Structure of the lesson :

1.1 Introduction

Often, we come across the study of physical systems subjected to periodic disturbances. In many cases, however, the forces such as torques, voltages or currents which act on a system, although periodic are by no means so simple as pure sine and cosine waves. Any complicated periodic function $f(x)$ of period 2π that appears in applications can be represented by a Fourier trigonometric series and formulae for the coefficients fo such series can easily be derived such that the series converges to $f(x)$. Later, the results to functions of arbitrary period can be extended.

Since many practical problems do not involve periodic functions, it is desirable to generalize the method of Fourier Series to include nonperiodic functions. Roughly speaking, if we start with a periodic function $f_T(x)$ of period T and let T approach infinity, then the resulting function $f(x)$ is no longer periodic or is of infinite period. Then $f(x)$ is represented by Fourier integral rather than Fourier Series.

15.2 Periodic Functions :

A function $f(x)$ is said to be periodic if it is defined for all real x and if there is some positive number T such that

$$f(x+T) = f(x) \text{ for all } x \text{ ------------- (1)}$$

The number T is then called a period of $f(x)$.

The graph of such a function is obtained by periodic repetition of its graph in any interval of length T.

From (1), it follows that, if n is any integer $f(x+nT) = f(x)$ for all x.

so that any integral multiple $nT(n \neq 0)$ is also a period.

Familiar and simple examples are sine and cosine functions The series which will arise in this connection will be of the form.

$$a_0 + (a_1 \cos x + a_2 \cos 2x + ...) + (b_1 \sin x + b_2 \sin 2x + ...) \text{ .... (2)}$$

where $a_0, a_1, a_2 ...., b_1, b_2, ...$ are real constants. Such a series is called a trigonometric series and the a_n and b_n are called the coefficients of the series. Each term of the seires(2) has the period 2π. Hence if the series converges, its sum will be a function of period 2π.

15.3 Definition of Fourier Series :

Let us suppose that $f(x)$ is a periodic function with period 2π which can be represented by a trigonometric series

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \quad \text{------------- (3)}$$

Given such a function $f(x)$, we determine the coefficients a_n and b_n in (3).

We first determine a_0. Integrating on both sides of (3) from $-\pi$ to $+\pi$, we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)\right]dx$$

If term by term integration is allowed, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\int_{-\pi}^{\pi} dx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,dx+b_n\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

The first term on the right equals to $2\pi a_0$, while all the other integrals are zero. Hence out first result is

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Which is nothing but the area under the curve of $f(x)$ from $-\pi$ to $+\pi$, divided by 2π.

Secondly, we determine $a_1, a_2, ...a_n$ by a similar procedure. We multiply (3) by $\cos mx$ where m is any fixed positive integer and then integrate from $-\pi$ to π, finding

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)\right]\cos mx\;dx \quad \text{---------- (5)}$$

Term by term integration of equation (5) gives

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos nx\,dx+\sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos mx\cos nx\,dx+\sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\cos mx\;\sin nx\,dx$$

On the RHS, the first integral is zero and so is the last integral as the integrand is an odd function. In the second term,

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx$$

$$= \quad 0 \quad + \quad 0 \qquad \text{for } n \neq m$$

$$= \quad 0 \quad + \quad \frac{2\pi}{2} \qquad \text{for } n = m$$

$\therefore$ (5) becomes $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \, dx$ ------------------ (6)

m = 1, 2, ...

Lastly, we determine $b_1, b_2, ...$ in equation (3). If we multip;y (3) by $\sin mx$ where m is any fixed positive integer and then integrate from $-\pi$ to π, we have,

$$\int_{-\pi}^{\pi} f(x)\sin mx \, dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\sin mx \, dx$$

$$= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx \quad \text{-------(7)}$$

On the RHS of (7), the first integral is zero and the next integral vanishes as the integrand is odd.

In the third term, $\int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx$

$$= \quad 0 \quad - \quad 0 \quad \text{when } n \neq m$$

$$= \frac{1}{2}2\pi \quad - \quad 0 \quad \text{when } n = m$$

$\therefore$ (7) gives us

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx \, dx \qquad m = 1, 2...... \quad (8)$$

Writing n in place of m in (6) and (8),

we have altogether the <u>Euler Formulae</u>

$$\left.\begin{array}{ll} (a) & a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ (b) & a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \\ (c) & b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx \\ & \qquad n = 1, 2..... \end{array}\right\} \text{--------(9)}$$

The series (3) is then called Fourier seires corresponding to $f(x)$ and its coefficients obtained from (9) are called Fourier Coefficients of $f(x)$.

***Note :** Because of the Periodicity of the integrands, the interval of integration is (9) may be replaced by any other interval of length 2π.*

With some mathematical rigour, as laid down by Dirichlet's conditions the definition of Fourier series can be stated as follows :

' If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left and right hand derivative at each point of the interval, there the corresponding Fourier Series (3) with coefficients given by (9) is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and right handed limits of $f(x)$ at x_0.

Thus, if the Fourier series corresponding to a function $f(x)$ converges to $f(x)$, the series will be called the Fourier series of $f(x)$. Then we say that $f(x)$ is represented by Fourier series.

Example (1) :

Find the Fourier series of the periodic function $f(x)$ given by

$$f(x) = \left.\begin{matrix} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{matrix}\right\} \text{ and } f(x+2\pi) = f(x)$$

(Functions of this type may occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.,)

Solution :

Let the Fourier series be

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \text{ ----------- (3)}$$

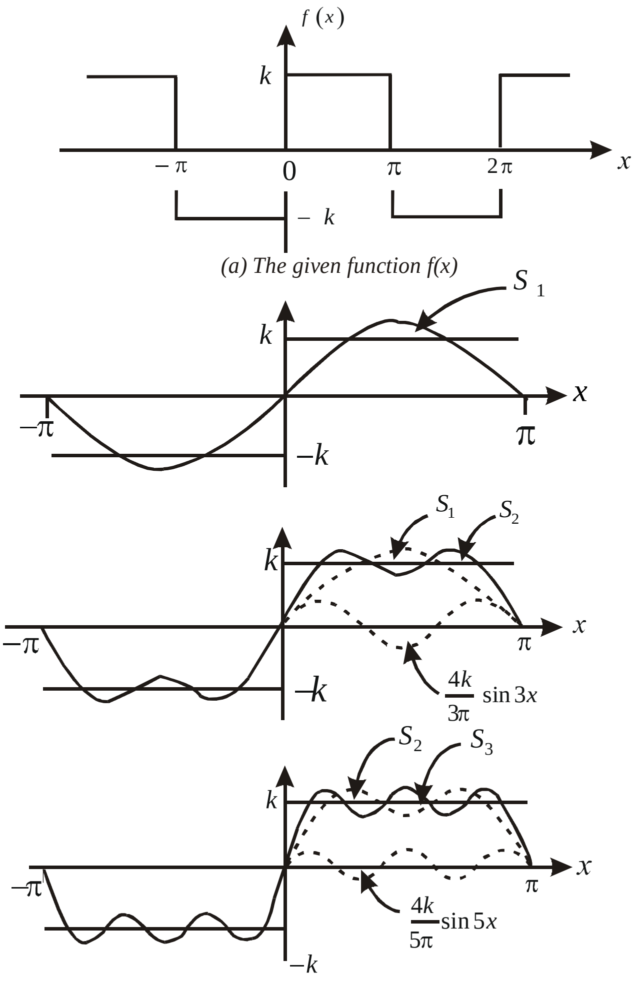
where

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned}\right\} \text{------(9)}$$

Then $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} -k\, dx + \int_{0}^{\pi} k\ dx \right]$

$$= \frac{1}{2\pi} [-k\pi + k\pi] = 0$$

or the area under the curve of $f(x)$ between $-\pi$ and π as can be seen in Fig. 1(a) is zero and thereby $a_0 = 0$.



(b) The first three partial sums of the corresponding Fourier series

Fig. 1 of Example 1

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\, dx + \int_{0}^{\pi}(k)\cos nx\, dx\right]$$

$$= \frac{k}{\pi}\left\{\left[-\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{k}{\pi}\{0 + 0\} = 0$$

Similarly $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\, dx + \int_{0}^{\pi} k \sin nx\, dx\right]$$

$$= \frac{k}{\pi}\left[-\int_{0}^{\pi}\sin(-nx)\,dx + \int_{0}^{\pi}\sin nx\, dx\right]$$

$$= \frac{2k}{\pi}\int_{0}^{\pi}\sin nx\, dx = \frac{2k}{\pi}.\ \left.\frac{-\cos nx}{n}\right|_{0}^{\pi}$$

$$= -\frac{2k}{\pi n}(\cos n\pi - 1) = \frac{2k}{n\pi}\left[1-(-1)^n\right]$$

(i.e.) $b_1 = \frac{4k}{\pi};\ b_2 = 0;\ b_3 = \frac{4k}{3\pi},\ b_4 = 0, ...$

(i.e.) All b's with even suffixes are zero.

So the Fourier series(3) for the given function is

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right)$$

Note : *The partial sums are*

$$S_1 = \frac{4k}{\pi}\sin x \qquad \textit{first term only}$$

$$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right) = \textit{ Sum to first two terms}$$

$$S_3 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right) = \textit{ Sum to first three terms.}$$

These partial sums are celarly depicted in the graph as shown in Fig. 1b for a better understanding of the convergence of the series. We notice that at $x=0$ *and* π*, the points of discontinuity of* $f(x)$ *(Fig 1a), all partial sums have the value zero (Fig. 1b) as it is the arithmetic mean of the values of* $f(o_-)$ *and* $f(o_+)$ *for x = 0 and* $f(\pi_-)$ *and* $f(\pi_+)$ *for* $x=\pi$ *respectively. Furthermore, at* $x=\frac{\pi}{2}$*, the sum of the series is given by*

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5}.........\right)$$

or

$$\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - ....\right)$$

Example - 2

What is the Fourier expansion of the periodic function whose definition in one period is

$$f(x) = \left.\begin{matrix} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{matrix}\right\}$$

Hence show that

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + ...$$

Solution : Let the Fourier Series be as in equation (3) with the coefficients given by equation (9)

Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx+\int_{0}^{\pi}\sin x\,dx\right]$$

$$=\frac{1}{2\pi}\left[-\cos x\right]_0^{\pi}=\frac{2}{2\pi}=\frac{1}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ \cos nx\,dx+\int_{0}^{\pi}\sin x\ \cos nx\ dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{1}{2}\left\{\frac{\cos(1-n)x}{(1-n)}+\frac{\cos(1+n)x}{1+n}\right\}\right]_0^{\pi}$$

$$=-\frac{1}{2\pi}\left[\frac{\cos(\pi-n\pi)}{1-n}+\frac{\cos(\pi+n\pi)}{1+n}-\left(\frac{1}{1-n}+\frac{1}{1+n}\right)\right]$$

$$=-\frac{1}{2\pi}\left(\frac{-\cos n\pi}{1-n}+\frac{-\cos n\pi}{1+n}-\frac{2}{1-n^2}\right)$$

$$=\frac{\cos n\pi+1}{\pi(1-n)^2}\qquad \text{for } n\neq 1$$

$$a_1=\frac{1}{\pi}\int_{0}^{\pi}\sin x\ \cos\ x\ dx=\left.\frac{\sin^2 x}{2\pi}\right|_0^{\pi}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0} 0.\ \sin nx\ dx+\frac{1}{\pi}\int_{0}^{\pi}\sin x\ \sin nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{1}{2}\left(\frac{\sin(1-n)x}{1-n}-\frac{\sin(1+n)x}{1+n}\right)\right]_0^{\pi}=0\qquad \text{for } n\neq 1.$$

$$b_1=\frac{1}{\pi}\int_{0}^{\pi}\sin^2 x\,dx=\frac{1}{\pi}\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\pi}=\frac{1}{2}$$

$\therefore$ The required Fourier series is

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + ...\right)$$

If we put $x = \frac{\pi}{2}$, then

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left(-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \frac{1}{63} - ....\right)$$

or $$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + ...$$

15.4 Fourier Series for even and odd functions :

We know that a function $y = g(x)$ is said to be <u>even</u> if $g(-x) = g(x)$ for all x. The graph of such a function is symmetric with respect to the y-axis.

A function $h(x)$ is said to be <u>odd</u> if $h(-x) = -h(x)$ for all x

The function $\cos nx$ is even, while $\sin nx$ is odd function of x.

If $g(x)$ is an even function, then

$$\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx \qquad (g \text{ even}) \text{ ------------- } (10)$$

If $h(x)$ is an odd function, then

$$\int_{-a}^{a} h(x)\,dx = 0 \qquad (h \text{ odd}) \text{ -------------- } (11)$$

It is obvious that the product $q = gh$ of an even function g and an odd function h is odd.

Then the Fourier series of an even periodic function $f(x)$ having period 2π is a <u>Fourier Cosine Series</u>.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (f \text{ even}) \text{ -------------- } (12)$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2..... \qquad \text{------- (13)}$$

The Fourier series of an odd periodic function $f(x)$ having period 2π is a <u>Fourier Sine Series.</u>

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (f \text{ odd}) \text{ ------------- (14)}$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \text{ ----------------- (15)}$$

Example (3) :

Show that $f(x) = x^2 \ (-\pi < x \le \pi),\ f(x+2\pi) = f(x)$ has the Fourier Series

$$f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - ....\right)$$

Hence show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \sum_{n=1} \frac{1}{n^2}$$

Solution : Since the given function is an even function, the Fourier series consists of only cosine series.

(i.e.,) $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (f \text{ even}) \text{ ---------- (12)}$$

with $$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad n = 1, 2, 3, ... \text{ ----------- (13)}$$

Now $$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\left\{\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}.2x\,dx\right\}$$

$$= \frac{-4}{\pi n}\left\{\left[x.\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx.1.dx\right\}$$

$$= \frac{4}{\pi n^2}\pi\,(-1)^n + 0 = (-1)^n\frac{4}{n^2}$$

$\therefore$ The Fourier series is

$$f(x)=\frac{\pi^2}{3}-\frac{4}{1^2}\cos x+\frac{4}{2^2}\cos 2x-\frac{4}{3^2}\cos 3x-......$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-........\right)$$

Putting $x=\pi$, then the above series becomes

$$f(\pi)=\pi^2=\frac{\pi^2}{3}-4\left(\cos\pi-\frac{1}{4}\cos 2\pi+\frac{1}{9}\cos 3\pi-..........\right)$$

or $$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+.....=\sum_{n=1}\frac{1}{n^2}$$

15.5 Function of Arbitrary Period :

Suppose that $f(t)$ has an arbitrary period T. Let us introduce a new variable x such that $f(t)$, as a function of x, has period 2π .

Put $t=\frac{T}{2\pi}x$ so that $x=\frac{2\pi}{T}t$ ------------ (14)

Then $x=\pm\pi$ corresponds to $t=\pm\frac{T}{2}$, which means that f , as a function of x, has period 2π .

Therefore Fourier Series of the form

$$f(t)=f\left(\frac{T}{2\pi}x\right)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$ ----------- (15)

can be obtained.

The Coefficients can be derived from (9) in the form

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{2\pi} x\right) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{2\pi} x\right) \cos nx \; dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{2\pi} x\right) \sin nx \; dx$$

We could use these formulas directly, but the change to t simplifies calculation. Since

$x = \frac{2\pi}{T} t$, we have $dx = \frac{2\pi}{T} dt$,

and the interval of integration corresponds to the interval

$$-\frac{T}{2} \leqq t \leqq \frac{T}{2}.$$

Consequently, we obtain the Euler formulas

(a) $a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$

(b) $a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2n\pi t}{T} dt$ --------- (16)

(c) $b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2n\pi t}{T} dt$ $n = 1, 2, \ldots\ldots$

for the Fourier coefficients of $f(t)$. The Fourier series (15) with x expressed in terms of t becomes

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{T} t + b_n \sin \frac{2n\pi}{T} t \right) \quad \text{----------- (17)}$$

The interval of integration in (16) may be replaced by any interval of length T, for example, by the interval $0 \leqq t \leqq T$.

We now obtain Fourier series of an even function $f(t)$ having period T as a Fourier cosine series.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T}t \qquad (f \text{ even}) \text{ ----------- } (18)$$

with coefficients

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt, \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2n\pi}{T}t\,dt, \; n = 1, 2 ...... \text{ ------ } (19)$$

Similarly, the Fourier series of an odd function $f(t)$ having period T is a Fourier sine series.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi}{T}t \qquad (f \text{ odd}) \text{ -------------- } (20)$$

with Coefficients

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2n\pi}{T}t\,dt \qquad \text{--------------- } (21)$$

Example - 4 : Find the Fourier Series of the function

$$f(t) = \begin{cases} 0 & \text{when } -2 < t < -1 \\ k & \text{when } -1 < t < 1 \\ 0 & \text{when } \quad 1 < t < 2 \end{cases} \qquad T = 4.$$

Solution :

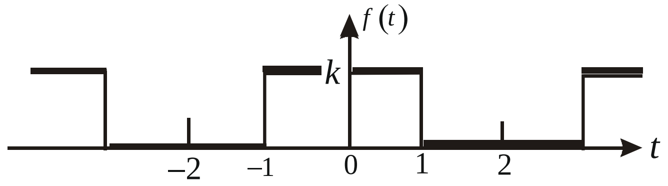


Fig. 2 $f(t)$ of the example 4.

Since f is even, $b_n = 0$. From (19),

$$a_0 = \frac{1}{2}\int_0^2 f(t)\,dt = \frac{1}{2}\int_0^1 k\ dt = \frac{k}{2}$$

$$a_n = \int_0^2 f(t)\cos\frac{n\pi}{2}t\ dt = \int_0^1 k\cos\frac{n\pi}{2}\ t\ dt = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

Thus $a_n = 0$ when n is even, $a_n = \frac{2k}{n\pi}$ when $n = 1, 5, 9, \ldots.$, and $a_n = \frac{-2k}{n\pi}$ when $n = 3, 7, 11, \ldots..$

Hence

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - + \ldots\ldots.\right)$$

Example - 5 : (Half-wave rectifier). A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave (Fig. 3). Develop the resulting periodic function

$$u(t) = \begin{cases} 0 & when \quad -T/2 < t < 0 \\ E\sin\,\omega t & when \quad 0 < t < T/2, \end{cases} \qquad T = \frac{2\pi}{\omega}$$

in a Fourier series.

Solution : Since $u = 0$ when $-T/2 < t < 0$, we obtain from (16a)

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\,dt = \frac{E}{\pi}$$

and from (16b) with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\ \sin\omega t\ \cos n\omega t\,dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

When $n = 1$, the integral on the right is zero, and when $n = 2, 3, \ldots..$

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right)$$

When n is odd, this is equal to zero, and for even n we obtain

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n=2,4,....)$$

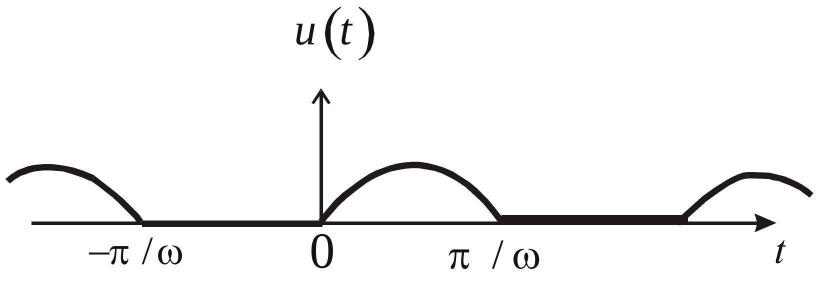


Fig. 3. Half-wave rectifier

In a similar fashion we find from (16c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2,3,....,$ Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right)$$

15.6 Half-Range expansions :

Let $f(t)$ have period $T = 2l$. If f is even, we obtain (18) and (19) the Fourier cosine series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}t \qquad (f \text{ even}) \text{ ------------- } (22)$$

with coefficients

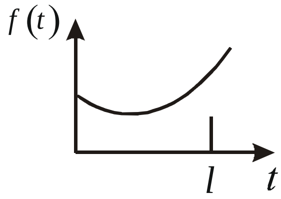
$$a_0 = \frac{1}{l}\int_0^l f(t)\,dt \qquad a_n = \frac{2}{l}\int_0^l f(t)\cos\frac{n\pi}{l}t\,dt \qquad n = 1,2,.... \text{ ------------ } (23)$$

Similarly, if f is odd, we obtain the Fourier sine series

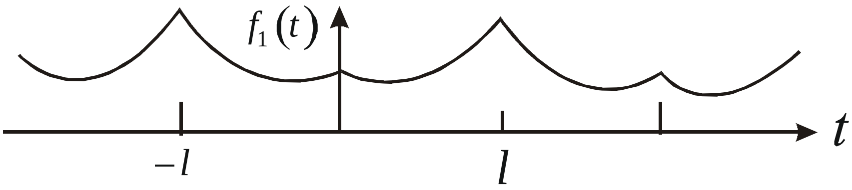
$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}t \qquad (f \text{ odd}) \text{ ----------- } (24)$$

with coefficients

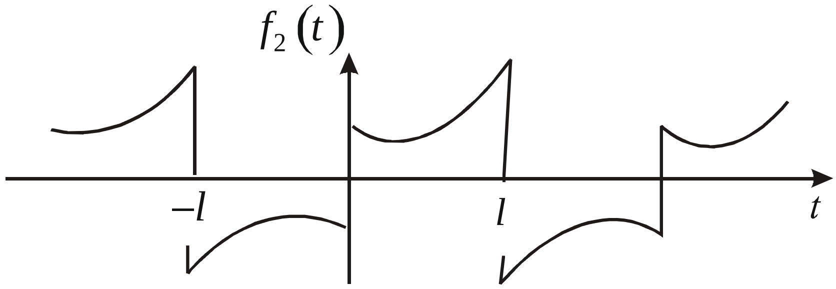
$$b_n = \frac{2}{l}\int_0^l f(t)\sin\frac{n\pi}{l}t\,dt. \qquad n = 1,2,..... \text{ ---------- } (25)$$



(a) The given function $f(t)$



(b) $f(t)$ continued as an even periodic function of period $2l$.



(c) $f(t)$ continued as an odd periodic function of period $2l$.

Fig. 4 Periodic continuations or extensions.

Now (23), and (25) use only the values of $f(t)$ between $t=0$ and $t=l$. Hence, for a function $f(t)$ given only over this interval, we can form the series (22) and (24). If $f(t)$ satisfies the conditions for a function to be represented by Fourier series, both series will represent the given function in the interval $0<t<l$. Outside this interval the series (22) will represent the even periodic extension or continuation of f having period $T=2l$ (Fig. 4b) and (24) will represent the odd periodic continuation of f (Fig. 4c). The series (22) and (24) with coefficients given by (23) and (25) are called half-range expansions of the given function $f(t)$. They will have important applications in connection with partial differential equations.

Example - 6: Find the half-range expansions of the function

$$f(t)=\begin{cases}\frac{2k}{l}t & when \quad 0<t<\frac{l}{2}\\ \frac{2k}{l}(l-t) & when \quad \frac{l}{2}<t<l\end{cases}$$

shown in Fig. 5

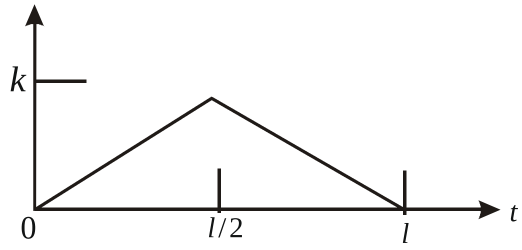


Fig. 5 The given function in Example 6.

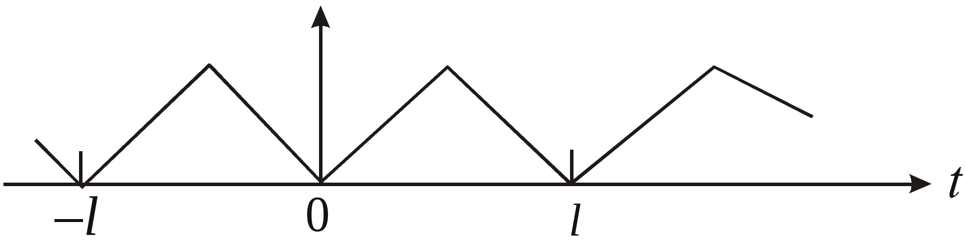
Solution : From (23), $a_0=\frac{1}{l}\left[\frac{2k}{l}\int_0^{l/2} t\,dt+\frac{2k}{l}\int_{l/2}^{l}(l-t)\,dt\right]=\frac{k}{2}$

$$a_n=\frac{2}{l}\left[\frac{2k}{l}\int_0^{l/2} t\cos\frac{n\pi}{l}t\ dt+\frac{2k}{l}\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{t}t\ dt\right]$$

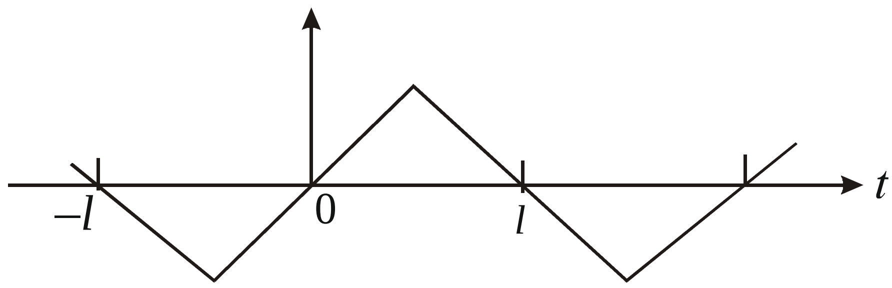
Now by integration by parts,

$$\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt=\frac{lt}{n\pi}\sin\frac{n\pi}{l}t\Big|_0^{l/2}-\frac{1}{n\pi}\int_0^{l/2}\sin\frac{n\pi}{t}t\,dt$$

$$=\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)$$



(a) Even continuation



(b) Odd continuation

Fig. 6. Periodic continuations of $f(t)$ in Example 6.

Similarly,

$$\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{l}t\,dt = -\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

By inserting these two results, we obtain

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

Thus,

$$a_2 = \frac{-16k}{2^2\pi^2},\ a_6 = \frac{-16k}{6^2\pi^2},\ a_{10} = \frac{-16k}{10^2\pi^2}\ ......$$

and $a_n = 0$ when $n \neq 2, 6, 10, 14, ...$ Hence the first half-range expansion of $f(t)$ is

$$f(t) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{l}t + \frac{1}{6^2}\cos\frac{6\pi}{l}t + ...\right)$$

This series represents the even periodic continuation of $f(t)$ shown in Fig. (6a)

Similarly, from (25),

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \text{----------- (26)}$$

and the other half-range expansion is

$$f(t)=\frac{8k}{\pi^2}\left(\frac{1}{l^2}\sin\frac{\pi}{l}t-\frac{1}{3^2}\sin\frac{3\pi}{l}t+\frac{1}{5^2}\sin\frac{5\pi}{l}t-+.........\right)$$

This series represents the odd periodic continuation of $f(t)$ shown in Fig. (6b).

Example - 7 : Find the half-range expansion of the function $f(t)=t-t^2 \qquad 0<t<1$

Solution : (i) The half-range cosine expansion is obtained by first extending $t-t^2$ from the given interval (0, 1) to the interval (-1, 0) by reflection in the y-axis and then taking the function thus defined from -1 to +1 as one period of a periodic function of period $2p=2$. (Once we understand the reasoning underlying the procedure we need give no thought to the extension but can write immediately).

$$\therefore b_n=0$$

$$a_n=\frac{2}{1}\int_0^1\left(t-t^2\right)\cos\frac{n\pi t}{1}\,dt$$

$$=2\left[\left(\frac{\cos n\pi t}{n^2\pi^2}+\frac{t}{n\pi}\sin n\pi t\right)-\left(\frac{2t}{n\pi^2}\cos n\pi t-\frac{2}{n^3\pi^3}\sin n\pi t+\frac{t^2}{n\pi}\sin n\pi t\right)\right]_0^1$$

$$=2\left(\frac{\cos n\pi-1}{n^2\pi^2}-\frac{2\cos n\pi}{n^2\pi^2}\right)$$

$$=-\frac{2\left(1+\cos n\pi\right)}{n^2\pi^2} \qquad n\neq 0$$

$$a_0=\frac{1}{1}\int_0^1\left(t-t^2\right)dt=\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_0^1=\frac{1}{6}$$

Hence it is possible to represent $f(t)=t-t^2$ for $0<t<1$ by the series

$$f(t)=\frac{1}{6}-\frac{4}{\pi^2}\left(\frac{\cos 2\pi t}{4}+\frac{\cos 4\pi t}{16}+\frac{\cos 6\pi t}{36}+.......\right) \text{ ------- (27)}$$

(ii). Similarly, the half-range sine expansion is obtained by first extending the given function $t-t^2$ to the interval (-1, 0) by reflection in the origin and then extending periodically the function thus defined over (-1, 1).

$$\therefore a_n = 0 \quad \text{and} \quad b_n = \frac{2}{1}\int_0^1 \left(t-t^2\right)\sin\frac{n\pi t}{1}\,dt$$

$$=2\left[\left(\frac{\sin n\pi t}{n^2\pi^2}-\frac{\cos n\pi t}{n\pi}.t\right)-\left(\frac{\sin n\pi t}{n^2\pi^2}.2t+\frac{2\cos n\pi t}{n^3\pi^3}-\frac{\cos n\pi t}{n\pi}t^2\right)\right]_0^1$$

$$=2\left[\left(\frac{-\cos n\pi}{n\pi}\right)-\left(\frac{2\left(\cos n\pi-1\right)}{n^3\pi^3}-\frac{\cos n\pi}{n\pi}\right)\right]=\frac{4\left(1-\cos n\pi\right)}{n^3\pi^3}$$

So it is also possible to represent $f(t)$ for $0<t<1$ by the series

$$f(t)=\frac{8}{\pi^3}\left(\frac{\sin \pi t}{1}+\frac{\sin 3\pi t}{27}+\frac{\sin 5\pi t}{125}+\ldots\ldots\right) \text{ -------- (28)}$$

(iii). (Series (27) and (28) are by no means the only Fourier series that will represent $t-t^2$ on the interval (0, 1). They are merely the most convinent or most useful ones. In fact, with every possible extension of $t-t^2$ from 0 to -1, there is associated a series yielding $t-t^2$ for $0<t<1$.)

A third series might be obtained by letting the extension as simply the function defined by $t-t^2$ it self for $-1<t<0$. In this case

$$a_n = \frac{1}{1}\int_{-1}^1 \left(t-t^2\right)\cos\frac{n\pi t}{1}\,dt$$

$$=\left[\left(\frac{\cos n\pi t}{n^2\pi^2}+\frac{\sin n\pi t}{n\pi}.t\right)-\frac{\cos n\pi t}{n^2\pi^2}.2t+\frac{2\sin n\pi t}{n^3\pi^3}-\frac{\sin n\pi t}{n\pi}t^2\right]_{-1}^1$$

$$=-\frac{4\cos n\pi}{n^2\pi^2} \qquad n\neq 0$$

$$a_0=\frac{1}{2.1}\int_{-1}^1\left(t-t^2\right)dt=\frac{1}{2}\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_{-1}^1=-\frac{1}{3}$$

and $b_n = \frac{1}{1}\int_{-1}^{1}\left(t - t^2\right)\sin\frac{n\pi t}{1}\, dt$

$$= \left[\left(\frac{\sin n\pi t}{n^2\pi^2} - \frac{\cos n\pi t}{n\pi}\, t\right) - \left(\frac{\sin n\pi t}{n^2\pi^2}.2t + \frac{2\cos n\pi t}{n^3\pi^3} - \frac{t^2\cos n\pi t}{n\pi}\right)\right]_{-1}^{1}$$

$$= -\frac{2\cos n\pi}{n\pi}$$

Hence, for $0 < t < 1$, it is also possible to write

$$f(t) = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi t}{1} - \frac{\cos 2\pi t}{4} + \frac{\cos 3\pi t}{9} - ....\right) + \frac{2}{\pi}\left(\frac{\sin \pi t}{1} - \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} - ......\right) \text{ ------ (29)}$$

15.7 Alternative Forms of Fourier Series :

The original form of the Fourier series (17) of a function can be connected into several other trigonometric forms and complex exponential forms.

Consider

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{T}t + b_n \sin\frac{2n\pi}{T}t\right) \text{ ---------- (17)}$$

with coefficients

$$\left.\begin{aligned} a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt; \quad a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi}{T}t\ dt \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\frac{\sin 2n\pi}{T}t\ dt \end{aligned}\right\} \text{---------(16)}$$

(Now we apply to each pair of terms of the same frequecy the usual procedure for reducing the sum of a sine and a cosine of the same angle to a single term).

Now (17) can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}}\cos\frac{2n\pi}{T}t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}}\sin\frac{2n\pi}{T}t\right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \left(\cos \gamma_n \cos \frac{2n\pi}{T} t + \sin \gamma_n \sin \frac{2n\pi}{T} t \right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos \left(\frac{2n\pi}{T} t - \gamma_n \right) \text{ -------- (30)}$$

or, equally well

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \left(\sin \delta_n \cos \frac{2n\pi}{T} t + \cos \delta_n \frac{\sin 2n\pi}{T} t \right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \sin \left(\frac{2n\pi}{T} t + \delta_n \right) \text{ ------- (31)}$$

where

$$A_0 = a_0$$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$\cos \gamma_n = \sin \delta_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}$$

$$\sin \gamma_n = \cos \delta_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}$$

$$\delta_n = \frac{\pi}{2} - \gamma_n$$

Here A_n is the resultant amplitude of the components of frequency $\frac{2n\pi}{T}$, that is the amplitude of the nth harmonic in (17). The phase angles γ_n and δ_n measure the lag or lead of the nth harmonic with reference to a pure cosine or pure sine wave of the same frequency.

Equation (30) and (31) are the representations of the Fourier series in terms of phase angles and resultant amplitudes.

Again the complex exponential form of the Fourier series is obtained by substituting the exponential equivalents of the cosine and sine terms into the series (17).

We know that

$$\cos\frac{2n\pi}{T}t = \cos\omega_n t = \frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2}$$

and $$\sin\frac{2n\pi}{T}t = \sin\omega_n t = \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2}$$

where $\omega_n = \frac{2n\pi}{T}$

$$\therefore a_n \cos\frac{2n\pi}{T}t + b_n \sin\frac{2n\pi}{T}t$$

$$= a_n \cos\omega_n t + b_n \sin\omega_n t$$

$$= a_n \frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} + b_n \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}$$

$$= \frac{a_n - ib_n}{2} e^{i\omega_n t} + \frac{a_n + ib_n}{2} e^{-i\omega_n t} \text{ ----------- (33)}$$

If we define

$$c_0 = a_0;\ c_n = \frac{a_n - ib_n}{2};\ c_{-n} = \frac{a_n + ib_n}{2}$$

then, in view of (33), equation (17) can be written in a more symmetric form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$$

where

$$c_n = \frac{a_n - i b_n}{2} = \frac{1}{2}\left[\frac{2}{T}\int_{-T/2}^{T/2} f(t)\left(\cos\omega_n t - i\sin\omega_n t\right) dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\omega_n t}\, dt$$

or

$$\left.\begin{array}{l} f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{2n\pi}{T}t} \\ \textit{where} \\ c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\frac{2n\pi}{T}t}\, dt \end{array}\right\} \text{ -------- (34)}$$

which is the required complex, exponential representation of the Fourier series.

15.8 The Fourier Integral :

Many practical problems do not involve periodic functions. Then it is desirable to generalize the method of Fourier series to include nonperiodic functions.

for example,

Consider the function

$$f_T(x) = \begin{cases} 0 & when \quad -T/2 < x < -1 \\ 1 & when \quad -1 < x < 1 \\ 0 & when \quad 1 < x < T/2 \end{cases}$$

having period $T > 2$. For $T \to \infty$ we obtain the function

$$f(x) = \lim_{T\to\infty} f_T(x) = \begin{cases} 1 & when \ -1 < x < 1 \\ 0 & otherwise \end{cases}$$

as we can see in Fig. 7.

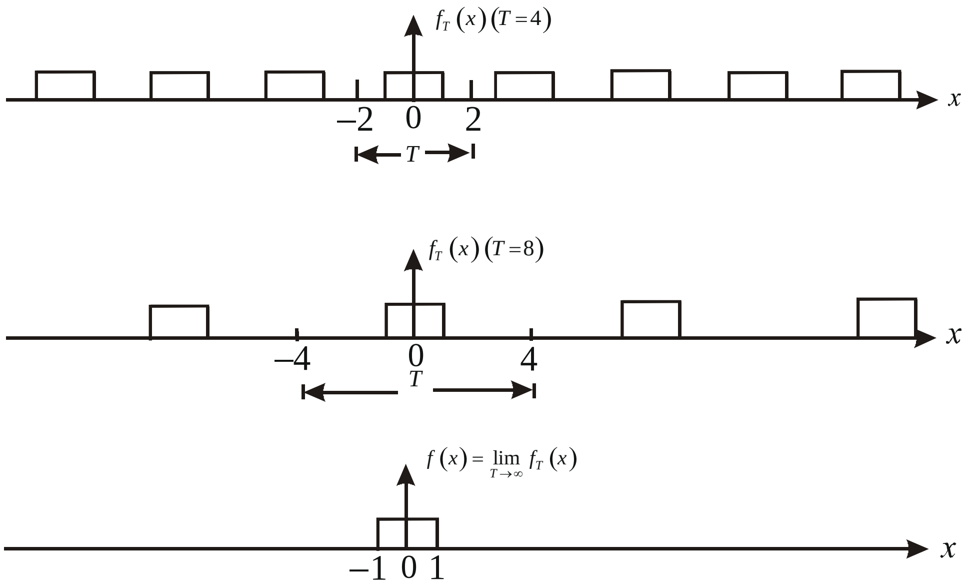


Fig. 7

By way of another example,

let $f_T(x) = e^{-|x|}$ when $-T/2 < x < T/2$ and $f_T(x+T) = f_T(x)$.

as given in fig. 8.

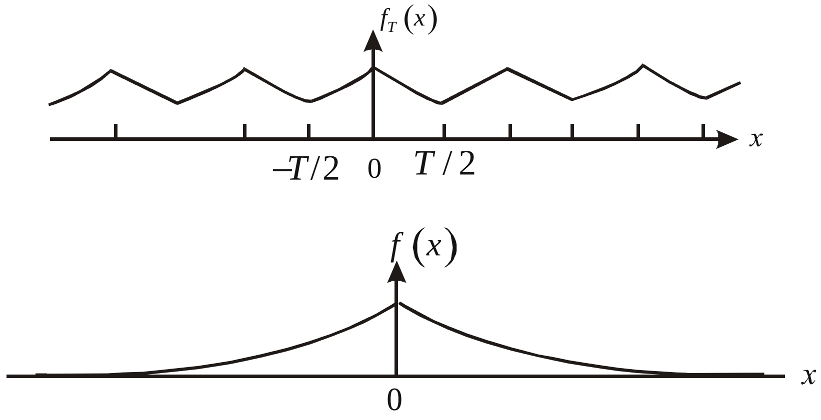


Fig. 8

Then $$f(x) = \lim_{T \to \infty} f_T(x) = e^{-|x|}.$$

These two functions as clearly depicted in Figs. 7 and 8 give us better understanding of the non-periodic behaviour of the functions in the limit.

This concept of the non-periodicity of the functions will lead to the integral representations of the Fourier series in the following way.

Now let us start from a periodic function $f_T(x)$ that has period T and can be represented by a Fourier series :

$$f_T(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{T}x + b_n \sin\frac{2n\pi}{T}x\right).$$

If we use the short notation

$$w_n = \frac{2n\pi}{T}$$

and insert a_n and b_n according to the Euler formulas (16), denoting the variable of integration by v, we obtain

$$f_T(x) = \frac{1}{T}\int_{-T/2}^{T/2} f_T(\mathrm{v})\, d\mathrm{v}$$

$$+ \frac{2}{T}\sum_{n=1}^{\infty}\left[\cos\omega_n x \int_{-T/2}^{T/2} f_T(\mathrm{v})\cos w_n \mathrm{v}\; dv + \sin w_n x \int_{-T/2}^{T/2} f_T(\mathrm{v})\sin\omega_n \mathrm{v}\; d\mathrm{v}\right]$$

Now,

$$w_{n+1} - w_n = \frac{2(n+1)\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T}$$

and we set

$$\Delta w = w_{n+1} - w_n = \frac{2\pi}{T}$$

Then $\frac{2}{T} = \frac{\Delta w}{\pi}$, and we may write that Fourier series in the form

$$f_T(x) = \frac{1}{T}\int_{-T/2}^{T/2} f_T(v)\,dv$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos(w_n x)\Delta w \int_{-T/2}^{T/2} f_T(v)\cos w_n v\,dv + \sin(w_n x)\Delta w \int_{-T/2}^{T/2} f_T(v)\sin w_n v\,dv\right] \text{---- (35)}$$

This representation is valid for any fixed T, arbitrarily large, but finite.

We now let T approach infiinity and assume that the resulting nonperiodic function

$$f(x) = \lim_{T\to\infty} f_T(x)$$

is absolutely integrable on the x- axis, that is, the integral

$$\int_{-\infty}^{\infty} |f(x)|\,dx \text{ ----------- (36)}$$

exists. Then $\frac{1}{T} \to 0$ and the value of the first term on the right side of (35) approaches zero. Furhtermore, $\Delta w = \frac{2\pi}{T} \to 0$ and it seems plausible that the infinite series in (36) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right] dw \text{ ----- (37)}$$

If we introduce the short notations

$$A(W) = \int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \int_{-\infty}^{\infty} f(v)\sin wv\,dv \text{ ------ (38)}$$

this may be written in the form

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right] dw \text{ --------- (39)}$$

which is a representation of $f(x)$ by a so-called Fourier integral. This approach merely suggests the representation (39) but by no means establishes it. Sufficient conditions for the validity of (39) are as follows.

" If $f(x)$ is pieceiwise continuous in every finite interval and has a right-and left-hand derivative at every point and the integral (36) exists, then $f(x)$ can be represented by a Fourier integral. At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left - and right - hand limits of $f(x)$ at that point."

If $f(x)$ is an even function then $B(\omega)=0$ in (38) and

$$A(w) = 2\int_0^{\infty} f(v)\cos\omega v \, dv \text{ --------- (40)}$$

$\therefore$ (39) reduces to the simpler form

$$f(x)=\frac{1}{\pi}\int_0^{\infty} A(\omega) \cos\omega x \, dx \qquad (f \text{ even}) \text{ --------- (41)}$$

If $f(x)$ is odd, then $A(\omega)=0$ and

$$B(\omega)=2\int_0^{\infty} f(v) \sin(\omega v) dv \text{ --------- (42)}$$

and (39) becomes

$$f(x)=\frac{1}{\pi}\int_0^{\infty} B(\omega)\sin\omega x \, d\omega \qquad (f \text{ odd}) \text{ -------- (43)}$$

If complex form of Fourier integral is required, write the compact form of (37) as

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos\left[\omega(x-v) dv\right]\right\} d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(\mathrm{v}) \frac{e^{i\omega(x-\mathrm{v})} + e^{-i\omega(x-\mathrm{v})}}{2} \, d\mathrm{v} \right\} d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(\mathrm{v}) \frac{e^{i\omega(x-\mathrm{v})}}{2} \, d\mathrm{v} \right\} d\omega + \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(\mathrm{v}) . \frac{e^{-i\omega(x-\mathrm{v})}}{2} \, d\mathrm{v} \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \left(\int_{-\infty}^{\infty} f(\mathrm{v}) . \, e^{-i\omega \mathrm{v}} \, d\mathrm{v} \right) d\omega$$

(after changing ω to $-\omega$ in the second term)

or

$$\left. \begin{aligned} & f(x) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} \, d\omega \\ & where \\ & g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathrm{v}) e^{-i\omega \mathrm{v}} \, d\mathrm{v} \end{aligned} \right\} \text{-------}(44)$$

which is called

'complex form of Fourier integral'

or

'Fourier integral pair'

or

'Fourier transform pair'

Example - 9 : Find the Fourier integral representation of the function (Fig. 9)

$$f(x) = \begin{cases} 1 & when \quad |x| < 1 \\ 0 & when \quad |x| > 1 \end{cases}$$ (single pulse)

Solution : From (38) we obtain

$$A(w) = \int_{-\infty}^{\infty} f(\mathrm{v}) \cos w\mathrm{v} \, d\mathrm{v} = \int_{-1}^{1} \cos w\mathrm{v} \, d\mathrm{v} = \left. \frac{\sin w\mathrm{v}}{w} \right|_{-1}^{1} = \frac{2 \sin w}{w} .$$

$$B(w) = \int_{-1}^{1} \sin w v \, d v = 0,$$

and (39) becomes

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw \text{ ----------- (45)}$$

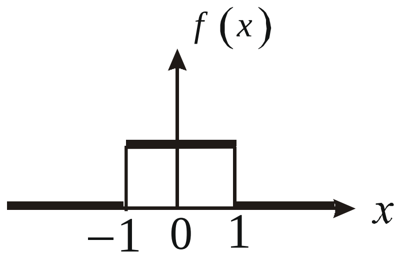


Fig. 9

The average of the left- and right-hand limits of $f(x)$ at $x=1$ is equal to $\frac{(1+0)}{2}$, that is $\frac{1}{2}$. Hence, from (45), we obtain the desired answer.

$$\int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw = \begin{cases} \frac{\pi}{2} & when \quad 0 \leqq x < 1, \\ \frac{\pi}{4} & when \quad x = 1 \\ 0 & when \quad x > 1 \end{cases}$$

we mention that this integral is called <u>Dirichlet's discontinuous factor.</u> Let us consider the case $x=0$, which is of particular interest. When $x=0$, then

$$\int_{0}^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2} \text{ ------ (46)}$$

we see that this integral is the limit of the so-called <u>sine integral</u>

$$Si(z) = \int_{0}^{z} \frac{\sin w}{w} dw \text{ ----- (47)}$$

as $z \to \infty$ (z real)

Example - (9) : Find the Fourier integral of

$$f(x) = e^{-kx} \text{ when } x>0 \text{ and } f(-x) = f(x) \qquad (k>0)$$

Solution : (cf. Fig. 8 where $k=1$). Since f is even, we have from (40)

$$A(w) = 2\int_0^{\infty} e^{-k\text{v}} \cos w\text{v}\, dv$$

Now, by integration by parts,

$$\int e^{-k\text{v}} \cos w\text{v}\, d\text{v} = -\frac{k}{k^2+w^2} e^{-k\text{v}} \left(-\frac{w}{k}\sin w\text{v} + \cos w\text{v}\right).$$

when $\text{v}=0$, the expression on the right equals $-\frac{k}{(k^2)+w^2}$; when v approaches infinity, it approaches zero because of the exponential factor. Thus

$$A(w) = \frac{2k}{k^2+w^2}$$ [or one can get this as $L(\cos\omega\text{v})$.]

and by substituting this in (41) we obtain the representation

$$f(x)=e^{-kx} = \frac{2k}{\pi}\int_0^{\infty} \frac{\cos wx}{k^2+w^2}\, dw \qquad (x>0,\ k>0)$$

or $$\int_0^{\infty} \frac{\cos wx}{k^2+w^2} dw = \frac{\pi}{2k} e^{-kx} \qquad (x>0,\ k>0)$$ ------ (48)

Similarly, from the Fourier integral (43) of the odd function

$$f(x)e^{-kx} \text{ when } x>0 \text{ and } f(-x)=-f(x) \quad (k>0).$$

we obtain the result

$$\int_0^{\infty} \frac{w\sin wx}{k^2+w^2}\, dw = \frac{\pi}{2} e^{-kx} \qquad (x>0,\ k>0)$$ ----------- (49)

The equation (48) and (49) are the so called Laplace integrals.

15.9 Orthogonal Functions and generalized Fourier Series :

Let $g_m(x)$ and $g_n(x)$ be two real functions which are defined on an interval $a \leqq x \leqq b$ and are such that the integral of the product $g_m(x), g_n(x)$ over that interval exists. We shall denote this integral by (g_m, g_n) and it is called as scalar or inner product of two functions $g_{m,} g_n$. Thus

$$(g_m, g_n) = \int_a^b g_m(x)\, g_n(x) dx \text{ ------- (50)}$$

The functions are said to be **orthogonal** on the interval $a \leqq x \leqq b$ if the integral (49) is zero, that is,

$$(g_m, g_n) = \int_a^b g_m(x)\, g_n(x) dx = 0 \qquad (m \neq n) \text{ -------- (51)}$$

The non-negative square root of (g_m, g_n) is called the norm of $g_m(x)$ and is generally denoted by $\|g_m\|$; thus

$$\|g_m\| = \sqrt{(g_m, g_n)} = \sqrt{\int_a^b g_m^2(x) dx} \text{ ------- (52)}$$

Clearly, an orthogonal set $g_1, g_2, ...$ on an interval $a \leqq x \leqq b$ whose functions have norm 1 satisfies the relations.

$$(g_m, g_n) = \int_a^b g_m(x) g_n(x) dx = \begin{cases} 0 & when \quad m \neq n \quad m = 1, 2, ..., \\ 1 & when \quad m = n \quad n = 1, 2, ..., \end{cases} \text{ -------- (53)}$$

such a set is called an orthonormal set of functions on the interval $a \leqq x \leqq b$.

Obviously, from an orthogonal set we may obtain an orthonormal set by dividing each function by its norm on the interval under consideration,

As an example, the functions $g_m(x) = \sin mx,\ m = 1, 2, ...$ form an orthogonal set on the interval $-\pi \leqq x \leqq \pi$, because

$$(g_m, g_n) = \int_{-\pi}^{\pi} \sin mx \sin nx\ dx = 0 \qquad (m \neq n) \text{ -------- (54)}$$

The norm $\|g_m\|$ equals $\sqrt{\pi}$, because

$$\|g_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \qquad (m=1,2,....)$$

Hence the corresponding orthonormal set consists of the functions

$$\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, ...$$

Another familiar orthogonal set is Legendre functions over the interval $(-1, 1)$.

Some important sets of real functions $g_1, g_2,.....$ occuring in applications are not orthogonal but have the property that for some functions $p(x)$,

$$\int_a^b p(x) g_m(x) g_n(x) dx = 0 \qquad \text{when } m \neq n \qquad \text{---------- (55).}$$

such a set is then said to be orthogonal with respect tot he weight function $p(x)$ on the interval $a \leqq x \leqq b$. The norm of g_m is now defined as

$$\|g_m\| = \sqrt{\int_a^b p(x) g_m^2(x) dx} \qquad \text{---------- (56)}$$

and if the norm of each function g_m is 1, the set is said to be orthonormal on that interval with respect to $p(x)$.

If we set $h_m = \sqrt{p}\, g_m$, then (55) becomes

$$\int_a^b h_m(x) h_n(x) dx = 0 \qquad (m \neq n).$$

that is, the functions h_m form an orthogonal set in the usual sense. Bessel, Laguerre, Hermite etc... functions belong to such set of orthogonal functions with respect to weight functions in an interval.

Looking at the derivation of the Euler formulas (9) for the Fourier coefficients, we see that we used merely the fact that the set of functions $\sin x$ or $\cos x$ is orthogonal on an interval of length 2π . This simple observation suggests the attempt to represent given functions $f(x)$ in terms of any other orthogonal set $g_1(x),\ g_2(x),....$ in the form

$$f(x)=\sum_{n=1}^{\infty} c_n\, g_n(x) = c_1\ g_1(x)+c_2\, g_2(x)+.... \text{ ------- (57)}$$

and determine the coeffocients $c_1,\ c_2,.....$. If the series (57) converges and represents $f(x)$, it is called a <u>generalized Fourier series</u> of $f(x)$, and its coefficients are called the Fourier constants of $f(x)$ with respect to that orthogonal set of functions.

To determine these constants, we multiply both sides of (57) by $g_m(x)$ and integrate over the interval $a \leqq x \leqq b$ on which the functions are orthogonal; assuming that term-by-term integration is permissible, we obtain

$$\int_a^b fg_m\, dx = \sum_{n=1}^{\infty} c_n \int_a^b g_n\, g_m\ dx$$

The integral for which $n=m$ is equal to the square of the norm $\|g_m\|$, while all the other integrals are zero because the functions are orthogonal. Thus,

$$\int_a^b fg_m\ dx = c_m \|g_m\|^2 \text{ ------------- (58)}$$

and the desired formula for the Fourier constants is

$$c_m = \frac{1}{\|g_m\|^2}\int_a^b f(x)\, g_m(x)\, dx \text{ ---- (59)}$$

15.10 Summary of the Lesson :

Starting with periodic functions, trigonometric Fourier series is developed and defined with Dirichlet's conditions. It is mathematically and graphically explained how far the partial sums and sum of the series converge to the actual value of the function. For even and odd periodic functions, compactness in the Fourier series is shown. Euler formulae for Fourier coefficients for the Fourier expansion of a periodic function with an arbitrary period have been derived. Half-range Fourier expansions are treated separately. Alternative forms of Fourier series are given. The Fourier integral as a limit of the sum of Fourier series is derived for non-periodic functions. Ultimately, this leads to the concept of Fourier transforms. Need based worked out examples are given. Finally the generalized Fourier series is defined with a stress on the basic concepts of orthogonal functions.

15.11 Key Terminology :

Periodic functions - Fourier series - Euler coefficients - Fourier series for arbitrary period - Half range expansions - Periodic extension - Phase angles - complex exponential series - Fourier integral - Dirichlet's discontinuous factor - Sine integral - Laplace integrals - Orthogonal functions - generalized Fourier series.

15.12 Self Assessment Questions :

1. Find the Fourier series for the periodic function $f(x)$ given by

$$f(x) = \left.\begin{matrix} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{matrix}\right\}$$

Hence prove that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$

(Hint : Sum the series at the discontinuity $x = 0$)

2. Obtain the half-range sine and cosine expansion of

$$f(t) = \left.\begin{matrix} t^2 & 0 < t < 1 \\ 2 - t & 1 < t < 2 \end{matrix}\right\}$$

3. Given the function $f(t) = \left.\begin{matrix} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{matrix}\right\}$

what is the amplitude of the resultant term of frequency $\frac{n\pi}{p}$ where p is half the period of $f(t)$ in its Fourier series. What is the phase of each of these terms relative to $\cos\frac{n\pi t}{p}$ and relative to $\sin\frac{n\pi t}{p}$

4. Find the complex form of the Fourier series of the periodic function $f(t) = \cos t \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

(Hint : Use the formula $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$)

5. Using Fourier integral representation, prove that

$$\int_0^\infty \frac{\cos ux}{u^2+1}\, du = \frac{\pi}{2} e^{-x} \quad x \geq 0$$

6. Find the Fourier integral representation of the functions

$$f(x) = \left.\begin{matrix} 0 & x<0 \\ \frac{1}{2} & x=0 \\ e^{-x} & x>0 \end{matrix}\right\}$$

verify the representation directly at the point $x=0$

15.13 Reference Books :

1. E. Kreyszig "Advanced Engineering Mathematics" Wiley Eastern Pvt., Ltd., 1971
2. C.R. Wylie Jr., "Advanced Engineering Mathematics" Wiley international Edition.
3. B.D. Gupta "Mathematical Physics" Vikas Publishing House, 1980.

Unit - IV

Lesson - 16

FOURIER TRANSFORMS

Objective of the lesson :

* To define infinite Fourier transform pair
* To know the relationship between Fourier and Laplace transforms
* To provide examples for better concepts
* To give the definition of finite Fourier cosine and sine transforms
* To apply the transforms for the solution of partial differential equations.

Structure of the lesson :

16.1 Introduction

In the last lesson, Fourier series expansions have been extensively studied which are appropriate for the analysis of periodic functions. Fourier transforms also perform a similar role in the analysis of functions which are not necessarily periodic. To be more elaborate, Fourier series allows a periodic functions to be represented as an infinite sum of harmonic oscillations at definite frequencies equal to multiples of the fundamental whereas the Fourier transform allows aperiodic function to be expressed as an integral sum over a continuous range of frequencies. With a suitable use of Dirac delta functions, the Fourier transform may be used to cover both periodic and aperiodic functions. The Fourier series then comes to be regarded as a special case of the Fourier transform.

Fourier transform are already introduced in the last lesson by way of Fourier integrals. In this lesson, importance has been given to Fourier transform (infinite), their properties and the examples and subsequently to finite Fourier transforms.

16.2 Nomenclature and definition of Fourier transforms :

There is no universally accepted convention governing the definitions of the terms 'Fourier transform' and 'inverse transform'. We shall adopt the notation below using t and ω as the variables.

$$f(\omega) = \int_{-\infty}^{\infty} F(t) e^{-i\omega t}\, dt \qquad \text{---------- (1)}$$

= Fourier transform of $F(t)$

$$= T\{F(t)\}$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega) e^{i\omega t}\, d\omega \qquad \text{---------- (2)}$$

= inverse Fourier transform of $f(\omega)$

$$= T^{-1}\{f(\omega)\}$$

Equations (1) and (2) together, will be called as 'Fourier Transform Pair'

'Fourier transforms' means 'infinite Fourier transforms', also called as 'complex Fourier transforms' .

For the sake of symmetry, the definition can also be choosen incorporating a factor $\frac{1}{\sqrt{2\pi}}$ in both transform (1) and inverse transform (2).

Symmetry can also be achieved by substituting $\omega = 2\pi\upsilon$

Different kind of definition arises due to some authors by taking the Fourier transform with a positive exponent while the inverse transform has a negative exponent.

we also define 'Fourier sine transform' or 'infinite sine transform' of $F(t)$ as

$$f_s(\omega) = \int_0^\infty F(t)\sin\omega t\ dt \qquad \text{----------------- (3)}$$

$= T_s\{F(t)\}$ = Fourier sine transform of $F(t)$

and

$$F(t) = \frac{2}{\pi}\int_0^\infty f_s(\omega)\sin\omega t\ d\omega \qquad \text{------------- (4)}$$

$$= T_s^{-1}\{f_s(\omega)\}$$

= Inverse Fourier sine transform of $f_s(\omega)$

Similarly, the Fourier cosine transforms or infinite Fourier cosine transform of $F(t)$ can be defined as

$$f_c(\omega) = \int_0^\infty F(t)\cos\omega t\ dt \qquad \text{------------- (5)}$$

= $T_c\{F(t)\}$ = Fourier cosine transform of $F(t)$.

and $$F(t) = \frac{2}{\pi}\int_0^{\infty} f_c(\omega)\cos\omega t\, d\omega \text{ ------------- (6)}$$

$$= T_c^{-1}\{f_c(\omega)\}$$

= Inverse Fourier cosine transform of $f_c(\omega)$.

Here also, for the sake of symmetry some authors may take $\sqrt{\frac{2}{\pi}}$ as the factors for (3) and (4); and for (5) and (6).

Throughout this lesson, we confine to the definition (1) to (6). Capital letters for the functions in transforms and lower case letters for inverse transforms will be used.

It is a cautionary note that one should invariably give the type of definition of transform pair that he is following before attempting any problem on Fourier transforms.

16.3 Relationship between Fourier transform and Laplace transform :

If we define the function $F(t)$ as

$$F(t) = \left.\begin{matrix} e^{-xt}\,G(t) & t>0 \\ 0 & t<0 \end{matrix}\right\}$$

then the Fourier transform of $F(t)$ is

$$T\{F(t)\} = \int_{-\infty}^{\infty} e^{-iyt}\, F(t)\, dt$$

$$= \int_{-\infty}^{0} e^{-iyt}\, 0\, dt + \int_0^{\infty} e^{-iyt}\, e^{-xt}\, G(t)\, dt$$

$$= \int_0^{\infty} e^{-(x+iy)t}\, G(t)\, dt$$

$$= \int_0^\infty e^{-st} G(t)\,dt \quad \text{where } s = x + iy$$

$$= L\{G(t)\}$$

which shows the relationship between Fourier and Laplace transforms.

16.4 Linearity Property :

If C_1 and C_2 are arbitrary constants, then

$$T\{C_1 F(t) + C_2 G(t)\} = C_1 T\{F(t)\} + C_2 T\{G(t)\}$$

Proof : $T\{C_1 F(t) + C_2 G(t)\}$

$$= \int_{-\infty}^{\infty} e^{-i\omega t}\left[C_1 F(t) + C_2 G(t)\right] dt$$

$$= C_1 \int_{-\infty}^{\infty} F(t) e^{-i\omega t}\,dt + C_2 \int_{-\infty}^{\infty} G(t) e^{-i\omega t}\,dt$$

$$= C_1 T\{F(t)\} + C_2 T\{G(t)\}$$

16.5 Scaling Property (Similarity Theorem) :

If $f(\omega)$ is the Fourier transform of $F(t)$, then $\frac{1}{a} f\left(\frac{\omega}{a}\right)$ is the Fourier transform of $F(at)$.

Proof : We know that the definition

$$T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$$

$$\therefore T\{F(at)\} = \int_{-\infty}^{+\infty} e^{-i\omega t} F(at)\,dt$$

$$T\{F(at)\} = \int_{-\infty}^{\infty} e^{-i\frac{\omega}{a}x} F(x)\frac{dx}{a} \qquad \text{Put } at = x \Rightarrow dt = \frac{dx}{a}$$

$$=\frac{1}{a} f\left(\frac{\omega}{a}\right) \text{ by definition.}$$

Note : *Corresponding results hold good for sine and cosine transforms.*

16.6 Time Shifting Proeprty :

If $\{F(t)\} = f(\omega)$, then $T\{F(t-a)\} = e^{-ia\omega} f(\omega)$

Proof : $T\{F(t-a)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F(t-a)\,dt$

$$= \int_{-\infty}^{\infty} e^{-i\omega(x+a)} F(x)\,dx$$

Put $t-a = x \Rightarrow dt = dx$

$$= e^{-ia\omega} \int_{-\infty}^{\infty} e^{-i\omega x} F(x)\,dx$$

$$= e^{-ia\omega} f(\omega)$$

16.7 Frequency Shifting Property :

If $T\{F(t)\} = f(\omega)$, then $T\left\{e^{i\omega_0 t} F(t)\right\} = f(\omega-\omega_0)$ ------------- (i)

Proof : $T\left\{e^{i\omega_0 t} F(t)\right\} = \int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega_0 t} F(t)\,dt$

$$= \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t} . F(t)\,dt$$

$$= f(\omega-\omega_0)$$

The result (i) of the theorem states that a shift of ω_0 in the frequency domain is equivalent to multiplication by $e^{i\omega_0 t}$ in the time domain. Or, the multiplication by a factor $e^{i\omega_0 t}$ translates the whole frequency spectrum $f(\omega)$ by an amount ω_0. Hence this tehorem is also known as the frequency - translation theorem.

In communication systems, this frequency translation is accomplished by multiplying a signal $F(t)$ by a sinusoidal signal, the process being known as modulation. Since a sinusoidal signal of frequency ω_0 can be expressed as the sum of exponentials, we have

$$F(t)\cos\omega_0 t = \frac{1}{2}\left[F(t)e^{i\omega_0 t} + F(t)e^{-i\omega_0 t}\right]$$

Then using frequency - shifting property (i),

We have

$$T\left\{F(t)\cos\omega_0 t\right\} = \frac{1}{2}\left[f(\omega+\omega_0)+f(\omega-\omega_0)\right] \text{ ---------- (ii)}$$

Similarly,

$$T\left\{F(t)\sin\omega_0 t\right\} = \frac{i}{2}\left[f(\omega+\omega_0)+f(\omega-\omega_0)\right] \text{ ------------ (iii)}$$

Thus the process of modulation translates the frequency spectrum by the amount of $\pm\omega_0$. This result is also known as 'modulation theorem'.

16.8 Time derivative property :

If $T\left\{F(t)\right\} = f(\omega)$, then $T\left\{F^{(n)}(t)\right\} = (i\omega)^n f(\omega)$ provided $F(t)$ and its derivatives of all orders tend to zero as $t \to \pm\infty$.

Proof : We have $T\left\{F(t)\right\} = f(\omega)$

$$\therefore T\left\{F'(t)\right\} = \int_{-\infty}^{\infty} e^{-i\omega t} F'(t)\,dt$$

$$= e^{-i\omega t} F(t)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\omega F(t)\, e^{-i\omega t}\,dt$$

$$= i\omega\, f(\omega) \text{ since } F(t) \text{ vanishes as } t \to \pm\infty$$

$$T\{F''(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F''(t)\,dt$$

$$= e^{-i\omega t} F'(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega) F'(t) e^{-i\omega t}\,dt$$

$$= 0 + i\omega T\{F'(t)\} \qquad \text{as } F'(t) = 0 \text{ at } t = \pm\infty$$

$$= (i\omega)^2 \; f(\omega)$$

$$\therefore T\{F^{(n)}(t)\} = (i\omega)^n \; f(\omega)$$

16.9 Frequency derivative property :

If $f(\omega) = T\{F(t)\}$ then $f^{(n)}(\omega) = T\{(-it)^n F(t)\}$

Proof : Given that $f(\omega) = T\{F(t)\}$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$$

So $$\frac{df(\omega)}{d\omega} = \frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\omega} e^{-i\omega t} \; F(t)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} (-it) F(t)\,dt$$

$$= T\{(-it) F(t)\}$$

$$f''(\omega) = \frac{d}{d\omega} f'(\omega) = \frac{d}{d\omega} T\{(-it) F(t)\}$$

$$= \frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-i\omega t} (-it) F(t) \, dt$$

$$= \int_{-\infty}^{\infty} (-it)^2 \; e^{-i\omega t} F(t) dt$$

$$= T\left\{(-it)^2 F(t)\right\}$$

$$\therefore \; f^{(n)}\omega = T\left\{(-it)^n F(t)\right\}.$$

16.10 Integration Property :

If $T\left\{F(t)\right\} = f(\omega)$, then $T\left\{\int_{-\infty}^{t} F(\lambda) d\lambda\right\} = \frac{1}{i\omega} f(\omega)$.

Proof : Let $G(t) = \int_{-\infty}^{t} F(\lambda) d\lambda \qquad \therefore G'(t) = F(t)$

So $T\left\{G'(t)\right\} = i\omega \; g(\omega)$ by time derivative property.

or $g(\omega) = \frac{1}{i\omega} T\left\{G'(t)\right\} = \frac{1}{i\omega} T\left\{F(t)\right\} = \frac{1}{i\omega} f(\omega)$

or $T\left\{\int_{-\infty}^{t} F(\lambda) \, d\lambda\right\} = \frac{1}{i\omega} f(\omega)$

16.11 Time Convolution Property :

Given two functions $F_1(t)$ and $F_2(t)$, the formation of the integral

$$F(t) = \int_{-\infty}^{\infty} F_1(\lambda) F_2(t-\lambda) d\lambda$$

$$= F_1(t)^* F_2(t)$$

is called the convolution integral of the two functions $F_1(t)$ and $F_2(t)$

Time convolution property says that

$$T\left\{F_1(t)^* F_2(t)\right\} = f_1(\omega)\ f_2(\omega)$$

Proof : $T\left\{F_1(t)^* F_2(t)\right\}$

$$= \int_{-\infty}^{\infty} e^{-i\omega t}\left(\int_{-\infty}^{\infty} F_1(\lambda) F_2(t-\lambda)\, d\lambda\right) dt$$

$$= \int_{-\infty}^{\infty} F_1(\lambda)\left(\int_{-\infty}^{\infty} e^{-i\omega t} F_2(t-\lambda)\, dt\right) d\lambda$$

$$= \int_{-\infty}^{\infty} F_1(\lambda)\left(\int_{-\infty}^{\infty} e^{-i\omega(\lambda+x)} F_2(x)\, dx\right) d\lambda \qquad \text{by putting } t-\lambda = x$$

$$= \left(\int_{-\infty}^{\infty} e^{-i\omega\lambda} F_1(\lambda)\, d\lambda\right)\left(\int_{-\infty}^{\infty} e^{-i\omega x} F_2(x)\, dx\right)$$

$$= f_1(\omega)\ f_2(\omega)$$

16.12 Frequency convolution property :

If $T\{F(t)\} = f(\omega)$, then $T\{F_1(t)\ F_2(t)\} = \frac{1}{2\pi} f_1(\omega)^* f_2(\omega)$

Proof : $\frac{1}{2\pi} f_1(\omega)^* f_2(\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\lambda)\, f_2(\omega-\lambda)\, d\lambda \qquad \text{(by convolution definition)}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\lambda)\left(\int_{-\infty}^{\infty} e^{-i(\omega-\lambda)t} F_2(t)\,dt\right)d\lambda$$ (by transform definition)

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\lambda)e^{i\lambda t}\int_{-\infty}^{\infty} e^{-i\omega t} F_2(t)\,dt\,d\lambda$$

$$=\int_{-\infty}^{\infty} e^{-i\omega t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\lambda)e^{i\lambda t}\,d\lambda\right)F_2(t)\,dt$$

$$=\int_{-\infty}^{\infty} e^{-i\omega t} F_1(t)\,F_2(t)\,dt$$ (by inverse transform definition)

$$=T\left\{F_1(t)\,F_2(t)\right\}$$

16.13 Parseval's Theorem :

If $T\{F(t)\} = f(\omega)$, then $\int_{-\infty}^{\infty} |f(\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty} |F(t)|^2 dt$

OR

The squared modulus of a function and the squared modulus of its transform have proportional areas. In particular, if ω is replaced by $2\pi\upsilon$, then $\int_{-\infty}^{\infty} |F(\upsilon)|^2\,d\upsilon = \int_{-\infty}^{+\infty} |F(t)|^2\,dt$.

Practically speaking, if each integral represents the energy associated with some process, the theorem says that the computation may be made in either the domain of the function or the domain of its transform.

Proof : We have the definition as

$$f(\omega) = T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$$

and $$F(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(\omega)\,d\omega$$

Let $F(t)$ and $G(t)$ are two functions (may be complex) whose Fourier transforms are $f(\omega)$ and $g(\omega)$, then

$$\int_{-\infty}^{\infty} f(\omega) g^*(\omega)\, d\omega = \int_{-\infty}^{\infty} f(\omega)\left(\int_{-\infty}^{\infty} G(t) e^{-i\omega t}\, dt\right)^* d\omega$$

$$= \int_{-\infty}^{\infty} G^*(t)\left(\int_{-\infty}^{\infty} f(\omega) e^{i\omega t}\, d\omega\right) dt$$

$$\int_{-\infty}^{\infty} G^*(t) 2\pi F(t)\, dt \quad \text{from the inverse transform}$$

$$= 2\pi \int_{-\infty}^{\infty} F(t) G^*(t)\, dt$$

In particular, if $F(t) = G(t)$, then

$$\int_{-\infty}^{\infty} |f(\omega)|^2\, d\omega = 2\pi \int_{-\infty}^{\infty} |F(t)|^2\, dt$$

which is the required Parseval's theorem.

16.14 Fourier transform of Dirac delta Function :

We know one of the properties of Dirac delta functions $\delta(t-a)$ as

$$\int_{-\infty}^{\infty} F(t)\delta(t-a)\, dt = F(a)$$

Now the Fourier transform of $\delta(t-a)$ is

$$T\{\delta(t-a)\} = \int_{-\infty}^{\infty} e^{-i\omega t}\, \delta(t-a)\, dt$$

$$= e^{-i\omega a} \qquad \text{(delta function property)}$$

In particular, if $a = 0$, then

$$T\{\delta(t)\} = 1$$

16.15 Examples :

(1) Find the Fourier transform of $F(t)$ defined by $F(t) = \left.\begin{matrix} 1 & |t|<a \\ 0 & |t|>a \end{matrix}\right\}$ and hence evaluate

(i) $\int_{-\infty}^{\infty} \frac{\sin \omega a \, \cos \omega t}{\omega} \, d\omega$, (ii) $\int_{0}^{\infty} \frac{\sin \omega}{\omega} \, d\omega$

Solution : As per the definition of transform,

$$T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F(t) dt = f(\omega)$$

$$= \int_{-\infty}^{-a} 0 e^{-i\omega t} dt + \int_{-a}^{a} 1.e^{-i\omega t} dt + \int_{a}^{\infty} e^{-i\omega t}.0 \, dt$$

$$= \int_{-a}^{a} e^{-i\omega t} dt = \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{-a}^{a} = \frac{1}{-i\omega}\left[e^{-i\omega a} - e^{i\omega a}\right]$$

$$= \frac{2}{\omega} \sin \omega a = f(\omega)$$

$$\therefore F(t) = T^{-1}\left[f(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{2. \sin \omega a}{\omega} d\omega$$

or $$\int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} e^{i\omega t} d\omega = \pi F(t)$$

(i.e.) $$\int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} (\cos \omega t + i \sin \omega t) d\omega = \left.\begin{matrix} \pi & |t|<a \\ 0 & |t|>a \end{matrix}\right\}$$

$$\int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} \cos \omega t \; d\omega + i \int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} \sin \omega t \; d\omega = \left.\begin{matrix} \pi & |t|<a \\ 0 & |t|>a \end{matrix}\right\}$$

Equating real parts on both sides

$$\int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} \cos \omega t \, d\omega = \left.\begin{matrix} \pi & |t| < a \\ 0 & |t| > a \end{matrix}\right\} \text{---------- (i)}$$

If $t=0$ and $a=1$, in the above integral

$$\int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} d\omega = \pi$$ $\because -a<t<a$ becomes $-1<t<1$ in which $t=0$ lies.

or $$\int_{0}^{\infty} \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2} \text{ ---------- (ii)}$$

(2) Find the Fourier transform of $F(t) = \begin{matrix} 1-t^2 & |t|<1 \\ 0 & |t|>1 \end{matrix}$ and hence evaluate $\int_0^\infty \frac{t\cos t - \sin t}{t^3} \cos \frac{t}{2} dt$

Solution : The Fourier transform of $F(t)$ is given by

$$f(\omega) = T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} F(t) dt$$

and its inverse is

$$F(t) = T^{-1}\{f(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f(\omega) d\omega$$

Now the given problem becomes

$$f(\omega) = \int_{-\infty}^{-1} 0 \, e^{-i\omega t} dt + \int_{-1}^{1} (1-t^2) e^{-i\omega t} dt + \int_{1}^{\infty} 0 \, e^{-i\omega t} dt$$

$$= \int_{-1}^{1} (1-t^2) e^{-i\omega t} dt$$

$$= (1-t^2) \frac{e^{-i\omega t}}{-i\omega} \Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-i\omega t}}{(-i\omega)} (-2t) dt$$

$$=0+\frac{2i}{\omega}\left\{\left[t.\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}-\int_{-1}^{1}\frac{e^{-i\omega t}}{-i\omega}1\,dt\right\}$$

$$=\frac{2i}{\omega}\left\{\frac{e^{-i\omega}}{-i\omega}-\frac{e^{+i\omega}}{i\omega}+\frac{1}{i\omega}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}\right\}$$

$$=-\frac{2}{\omega^2}\left(e^{i\omega}+e^{-i\omega}\right)+\frac{2i}{\omega^3}\left(e^{-i\omega}-e^{i\omega}\right)$$

$$=-\frac{4}{\omega^2}\cos\omega+\frac{4}{\omega^3}\sin\omega$$

$$=\frac{4}{\omega^3}\left(\sin\omega-\omega\cos\omega\right)=T\left\{F(t)\right\}$$

$$\therefore F(t)=T^{-1}f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{\omega^3}\left(\sin\omega-\omega\cos\omega\right)e^{i\omega t}\,d\omega$$

or $$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega-\omega\cos\omega}{\omega^3}\left(\cos\omega t+i\sin\omega t\right)d\omega=F(t)$$

Equating real parts on both sides

$$\int_{-\infty}^{\infty}\frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\omega t\,d\omega=\frac{\pi}{2}F(t)=\begin{cases}\frac{\pi}{2}\left(1-t^2\right) & |t|<1\\ 0 & |t|>1\end{cases}$$

Putting $t=\frac{1}{2}$ which lies in the interval $(-1,1)$,

we get $$\int_{-\infty}^{\infty}\frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\,d\omega=\frac{3\pi}{8}$$

$$2\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\,d\omega = \frac{3\pi}{8}$$

or $$\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\,d\omega = \frac{3\pi}{16}$$

(3) Obtain the Fourier transform of $F(t) = e^{-t^2/2}$ (Gaussian)

Solution : As per the definition,

$$f(\omega) = T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t}\, e^{-t^2/2}\,.\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\frac{-(t^2+2i\omega t)}{2}}\,dt = \int_{-\infty}^{\infty} e^{-\frac{(t+i\omega)^2}{2}}\,e^{\frac{-\omega^2}{2}}\,dt$$

$$= e^{-\frac{\omega^2}{2}}\int_{-\infty}^{\infty} e^{\frac{-(t+i\omega)^2}{2}}\,dt \qquad \text{Put } \frac{t+i\omega}{\sqrt{2}} = x \Rightarrow dt = \sqrt{2}\,dx$$

$$= \sqrt{2}\; e^{\frac{-\omega^2}{2}}\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= 2\sqrt{2}\; e^{\frac{-\omega^2}{2}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\sqrt{2}\; e^{-\frac{\omega^2}{2}}\,.\left|\frac{1}{2}\right.$$

$$= 2\sqrt{2}\,\frac{\sqrt{\pi}}{2}\,e^{-\omega^2/2} = \sqrt{2\pi}\,e^{-\omega^2/2} \qquad \text{(Gaussian)}$$

Note : *This problem can also be asked as "show that the Fourier transform of a Gaussian function is again a Gausian function".*

(4) Find the complex Fourier transform of $e^{-|t|}$

Solution : $f(\omega)=T\{F(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t}\, e^{-|t|}\, dt$

$$= \int_{-\infty}^{0} e^{-i\omega t}\, e^{-(-t)}\, dt + \int_{0}^{\infty} e^{-i\omega t}\, e^{-(+t)}\, dt$$

$$= \int_{0}^{\infty} e^{i\omega t}\, e^{-t}\, dt + \int_{0}^{\infty} e^{-(1+i\omega)t}\, dt$$

$$= \int_{0}^{\infty} e^{-(1-i\omega)t}\, dt + \int_{0}^{\infty} e^{-(1+i\omega)t}\, dt$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} \qquad \text{using } L\{1\}$$

$$= \frac{2}{1+\omega^2}$$

(5) Find the inverse Fourier transform of $f(\omega)=e^{-|\omega|k}$ where k belongs to $(-\infty,\infty)$.

Solution : Given that $f(\omega) = \left.\begin{matrix} e^{-(-\omega)k} & \omega \le 0 \\ e^{-(+\omega)k} & \omega \ge 0 \end{matrix}\right\}$

$$\therefore T^{-1} f(\omega) = F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, f(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{i\omega t}\, e^{\omega k}\, d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{i\omega}\, e^{-\omega k}\, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-(k+it)\omega}\, d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{-(k-it)\omega}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{k+it} + \frac{1}{k-it}\right]$$

$$= \frac{1}{2\pi} \frac{2k}{k^2+t^2} = \frac{1}{\pi} \frac{k}{k^2+t^2}$$

(6) Find the sine and cosine transform of $F(t) = 2e^{-5t} + 5e^{-2t}$

Solution : According to the definitions of cosine and sine transforms, we have

$$f_c(\omega) = T_c\{F(t)\} = \int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right)\cos\omega t \, dt \quad \text{--------- (i)}$$

and $$f_s(\omega) = T_s\{F(t)\} = \int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right)\sin\omega t \, dt \quad \text{---------- (ii)}$$

Equation (i) and (ii) are respectively the real and imaginary parts of the integral

$$\int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right) e^{i\omega t} \, dt$$

Now $$\int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right) e^{i\omega t} \, dt = \int_0^\infty \left[2e^{-(5-i\omega)t} + 5e^{-(2-i\omega)t}\right] dt$$

$$= \frac{2}{5-i\omega} + \frac{5}{2-i\omega} \qquad \text{from } L\{1\}$$

$$= \frac{2(5+i\omega)}{\omega^2+25} + \frac{5(2+i\omega)}{\omega^2+4} = 10\left(\frac{1}{\omega^2+25} + \frac{1}{\omega^2+4}\right) + i\left(\frac{2\omega}{\omega^2+25} + \frac{5\omega}{\omega^2+4}\right)$$

Thus $$\int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right)\cos\omega t \, dt = f_c(\omega) = 10\left(\frac{1}{\omega^2+25} + \frac{1}{\omega^2+4}\right) \quad \text{-------- (i)}$$

and $$\int_0^\infty \left(2e^{-5t} + 5e^{-2t}\right)\sin\omega t \, dt = f_s(\omega) = \left(\frac{2\omega}{\omega^2+25} + \frac{5\omega}{s^2+4}\right)$$

(7) Solve for $F(t)$ the integral equation

$$\int_0^\infty F(t)\sin\omega t\,dt = \left.\begin{matrix} 1 & 0\le\omega<1 \\ 2 & 1\le\omega<2 \\ 0 & \omega\ge 2 \end{matrix}\right\}$$

Solution : The LHS integral of the problem is the sine transform of $F(t)$ i.e., $f_s(\omega)$.

So $f_s(\omega) = \left.\begin{matrix} 1 & 0\le\omega<1 \\ 2 & 1\le\omega<2 \\ 0 & \omega\ge 2 \end{matrix}\right\}$ is given

we have to find the inverse of $f_s(\omega)$ as $F(t)$

So $F(t)=T_s^{-1}\left[f_s(\omega)\right] = \dfrac{2}{\pi}\displaystyle\int_0^\infty f_s(\omega)\sin\omega t\;d\omega$ -------- (4)

$$=\frac{2}{\pi}\int_0^1 1.\sin\omega t\,d\omega + \frac{2}{\pi}\int_1^2 2\sin\omega t\;d\omega$$

$$=\frac{2}{\pi}\left[-\frac{\cos\omega t}{t}\right]_{\omega=0}^1 + \frac{4}{\pi}\left[\frac{-\cos\omega t}{t}\right]_{\omega=1}^2$$

$$=-\frac{2}{\pi t}(\cos t-1)-\frac{4}{\pi t}(\cos 2t-\cos t)$$

$$=\frac{2}{\pi t}(1+\cos t-2\cos 2t)$$

(8) Find the Fourier cosine transform of e^{-t^2} (Gaussian)

Solution : According to the definition

$$f_c(\omega) = T_c\left\{F(t)\right\} = \int_0^\infty F(t)\cos\omega t\,dt$$

$$= \int_0^\infty e^{-t^2} \cos \omega t \, dt = I \quad \text{(say)}$$

$$\therefore \frac{dI}{d\omega} = -\int_0^\infty t e^{-t^2} \sin \omega t \; dt = +\frac{1}{2}\int_0^\infty -2t\, e^{-t^2} \sin\, \omega t . dt$$

Integrating by parts, we get

$$\frac{dI}{d\omega} = \frac{1}{2}\left\{\left[\sin \omega t . e^{-t^2}\right]_0^\infty - \int_0^\infty \omega\, e^{-t^2} \cos \omega t \; dt\right\}$$

$$= -\frac{\omega}{2}\int_0^\infty e^{-t^2} \cos \omega t \, dt = -\frac{\omega}{2} I$$

or $$\frac{dI}{I} = -\frac{1}{2}\omega \; d\omega$$

Integrating, $\log I = -\dfrac{\omega^2}{4} + \log k$

or $$I = k\, e^{-\omega^2/4}$$

when $\omega = 0$, then $I = \int_0^\infty e^{-t^2} \cos 0 \; dt = \int_0^\infty e^{-t^2} \; dt = \dfrac{\sqrt{\pi}}{2}$

$$\therefore \frac{\sqrt{\pi}}{2} = k.1 \quad \text{or} \quad k = \frac{\sqrt{\pi}}{2}$$

so $$I = \frac{\sqrt{\pi}}{2}\, e^{-\omega^2/4} \qquad \text{(Gaussian)}$$

(i.e.) The result is similar to the Example (3).

16.16 Finite Fourier Sine transform of *F(x)* :

The finite Fourier sine transform of $F(x)$ $0<0x<c$ is defined by

$$T_s\{F(x)\} = f_s(n)=\int_0^c F(x)\sin\frac{n\pi x}{c}dx \quad \text{------ (7)}$$

Where n is a positive integer.

Then the inverse finite Fourier sine transform of $f_s(n)$ (i.e.) $F(x)$ is given by

$$T_s^{-1}\{f_s(n)\}=F(x)=\frac{2}{c}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{c} \quad \text{-------- (8)}$$

For example, if we consider $F(x)=1$ in the interval $0<x<\pi$, then

$$f_s(n) = \int_0^{\pi} 1.\sin\frac{n\pi x}{\pi}\,dx \qquad \text{from (7)}$$

$$= \int_0^{\pi} \sin nx\,dx = \frac{1-(-1)^n}{n} \qquad (n=1,2,....)$$

If $f_s(n)$ for all n is substituted in (8) and sum the series we are supposed to get $F(x)$ as 1. (i.e.) There is only one function with a given transform or the inverse transform is unique. Considering another example $F(x)=x(0<x<\pi)$, we have

$f_s(n)=\int_0^{\pi} x\sin nx\,dx = \frac{\pi(-1)^{n+1}}{n}$ $(n=1,2,....)$. To find the inverse of this, substitute $f_s(n)$ with various values of n in equation (8) and that series should give us the value of $F(x)$ to be x.

16.17 Finite Fourier Cousine Transform of *F(x)* :

The finite Fourier cosine transform of $F(x)$ in the interval $0<x<c$ is defined by

$$T_c\{F(x)\} = f_c(n)=\int_0^c F(x)\cos\frac{n\pi x}{c}dx \quad \text{------ (9)}$$

where n is a positive integer.

Then the inverse finite Fourier cosine transform of $f_c(n)$ (i.e.) $F(x)$ is given by

$$T_c^{-1}\{f_c(n)\}=F(x)=\frac{1}{c}f_c(0)+\frac{2}{c}\sum_{n=1}^{\infty}f_c(n)\cos\frac{n\pi x}{c} \quad \text{------- (10)}$$

Both the formulae (8) and (10) can easily be understood from the Fourier series.

considering an example $F(x)=1$ in the interval $0<x<\pi$. We can have

$$f_c(n)=\int_0^{\pi}\cos\frac{n\pi x}{\pi}dx=\int_0^{\pi}\cos n\pi x\, dx$$

$$=\left.\frac{\sin n\pi x}{n\pi}\right|_0^{\pi}=0 \quad \text{for } n = 1, 2, ...$$

But $f_c(0)=\pi$. So finally $f(n)=\left.\begin{matrix}0 & n=1,2,.... \\ \pi & n=0\end{matrix}\right\}$

After substituting this in (10), it is obvious that

$$F(x)=\frac{1}{\pi}\pi=1$$

Note : *It is to be observed that through the functions $F(x)$ are simple, their finite Fourier sine or cosine transforms involve complexities.*

16.18 Some operational properties of finite sine and cosine transforms :

(i) Sine transform of $F'(x)$ in $0<x<c$ is given by

$$T_s\{F'(x)\}=\int_0^c F'(x)\sin\frac{n\pi x}{c}dx$$

$$=\left.F(x)\sin\frac{n\pi x}{c}\right|_0^c-\int_0^c F(x)\frac{\cos n\pi x}{c}\frac{n\pi}{c}dx$$

$$= 0 - \frac{n\pi}{c} f_c(n) = -\frac{n\pi}{c} T_c\{F(x)\}$$ ------------- (11)

(ii) If $F(x) = \int_0^x H(\lambda)\,d\lambda$, then the above property (i) takes the form as

$$T_s\{H(x)\} = -\frac{n\pi}{c} T_c\left\{\int_0^x H(\lambda)\,d\lambda\right\}$$ ---------- (12)

(iii) Sine transform of $F''(x)$ $0 < x < c$ is given by $T_s\{F''(x)\} = \int_0^c F''(x)\sin\frac{n\pi x}{c}dx$

$$= F'(x)\sin\frac{n\pi x}{\mathrm{c}}\Big|_0^c - \int_0^c F'(x)\cos\frac{n\pi x}{\mathrm{c}}\frac{n\pi}{c}dx$$

$$= -\frac{n\pi}{c}\int_0^c F'(x)\cos\frac{n\pi x}{c}dx$$

$$= -\frac{n\pi}{c}\left[F(x)\cos\frac{n\pi x}{c}\Big|_0^c + \int_0^c F(x)\sin\frac{n\pi x}{c}\frac{n\pi}{c}dx\right]$$

$$= -\frac{n\pi}{c}\left[(-1)^n F(c) - F(0) + \frac{n\pi}{c} f_s(n)\right]$$

$$-\left(\frac{n\pi}{c}\right)^2 f_s(n) + \frac{n\pi}{c}\left[F(0) - (-1)^n F(c)\right]$$ ------------ (13)

(9) If the Fourier sine transform of $F(x)$ is

$$f_s(n) = \frac{1-\cos n\pi}{n^2\pi^2} \qquad (n = 1, 2, ...)$$

determine $F(x)$ in the interval of $0 < x < \pi$

Solution : We know the inverse sine transform as

$$F(x) = \frac{2}{c}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{c} \text{ ------------ (8)}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{\pi} \text{ in the present problem}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2\pi^2}\sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2}\sin nx$$

which is the required result.

(10) Determine the function $F(x,y)$ such that $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad 0 < x < \pi$

with the boundary condition

F = 0 when $x = 0, \pi$

= 0 when $y = 0$

= F_0 (constant) when $y = \pi$

Solution : Since the interval is finite, let us use finite Fourier sine transform on both sides of the differential equation. Then

$$T_s\{F_{xx}\} + T_s\{F_{yy}\} = 0 \text{ ---------- (14)}$$

According to equation (13), in the interval $0 < x < \pi$ $\quad T_s\{F_{xx}\} = -n^2 f_s(n) + n[0-0]$

using boundary conditions $= -n^2 f_s(n)$

$\therefore$ 14 becomes $-n^2 f_s(n) + \int_0^{\pi} \frac{\partial^2 F}{\partial y^2}\sin nx\, dx = 0$

or $\quad \frac{\partial^2}{\partial y^2}\left[\int_0^{\pi} F(x,y)\sin nx\, dx\right] = n^2 f_s(n)$

(i.e.) $$\frac{\partial^2 f_s(n)}{\partial y^2} = n^2 f_s(n)$$

Its general solution is

$$f_s(n) = A\sinh(ny) \text{ --------- } (15)$$

At $y=\pi$, $f_s(n) = \int_0^\pi F(x,y)\sin nx\, dx$

$$= \int_0^\pi F_0 \sin nx\, dx$$

$$= F_0 \left.\frac{-\cos nx}{n}\right|_0^\pi$$

= 0 when n is even

$= -\dfrac{2F_0}{n}$ when n is odd.

So, when n is odd, at $y=\pi$, (15) becomes

$$-\frac{2F_0}{n} = A\sin h(n\pi) \qquad \therefore A = -\frac{2F_0}{n}\frac{1}{\sin h(n\pi)}$$

$\therefore$ (15) takes the form

$$f_s(n) = -\frac{2F_0}{n}\frac{\sin h(ny)}{\sin h(n\pi)} \qquad \text{when } n \text{ is odd.}$$

Hence the inversion formula (8) will give on replacing n by $2m+1$.

$$F(x,y) = -\frac{4F_0}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\frac{\sin h\overline{(2m+1)y}}{\sinh\overline{\pi(2m+1)}}\sin\overline{(2m+1)x}$$

which is the required result.

(11) Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with the boundary conditions.

$U(0,t)=0, U(\pi,t)=0, U(x,0)=2x$ when $0<x<\pi,\ t>0$.

Give physical interpretation of the problem.

Solution : Taking the finite Fourier sine transform on both sides of the differential equation,

We have $\int_0^\pi U_t(x,t)\sin nx\,dx = \int_0^\pi U_{xx}^{(x,t)}\sin nx\,dx$ ----------- (16)

Applying equation (13) for the interval $(0,\pi)$ to the RHS of (16), we get

LHS = $\int_0^\pi \frac{\partial \mathrm{u}(x,t)}{\partial \mathrm{t}}\sin x\,dx = \frac{\partial}{\partial t}\left[\int_0^\pi U(x,t)\sin x\,dx\right]$

$=n^2 u_s(n,t)$ (from RHS using the boundary condition)

or $\frac{\partial}{\partial t}u_s(n,t) = n^2 u_s(n,t)$

Integrating w.r.t. t, we have

$u_s(n,t) = Ae^{-n^2 t}$ ------ (17) Where the constant A is to be determined

or $\int_0^\pi U(x,0)\sin nx\,dx = Ae^{-n^2 t}$

At $t=0$, $\int_0^\pi U(x,0)\sin nx\,dx = Ae^0$

using the boundary condition that $U(x,0)=2x$

we have $\int_0^\pi 2x\sin nx\,dx = A$

or $$A = 2x \frac{(-\cos nx)}{n}\Bigg|_{x=0}^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} 2\,dx$$

$$= -\frac{2\pi}{n}\cos n\pi$$

$\therefore (17)$ becomes $u_s(n,t) = \frac{-2\pi}{n} = -\frac{2\pi}{n}\cos n\pi\, e^{-n^2 t}$

Applying the invension formula for finite Fourier sine transform, we have

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}\cos n\pi\; e^{-n^2 t}\right)\sin nx$$

For physical interpretation, $U(x,t)$ may be regarded as representing temperature of a solid at any point x at an instant of time t in a solid bounded by the planes $x=0$ and $x=\pi$. The boundary conditions $U(0,t)=0$ and $U(\pi,t)=0$ give the zero temperature at the end points of the solid. $U(x,0)=2x$ represents the initial temperature which is again a function of x.

16.19 Summary of the Lesson :

Definitions of infinite Fourier transforms with different nomenuclatures are given, several properties have been explained with sufficient details whenever needed. Convolution integral over a different integral $(-\infty \text{ to } \infty)$ is defined and accordingly convolution theorem is proved several examples on inifinite Fourier transfers, inifinite Fourier cosine and sine transforms are worked. While dealing with infinite Fourier cosine and sine transforms, generally it is convinient to make use of the concepts of Laplace transforms.

Definition of finite Fourier cosine and sine trasnsforms and their inverses are introduced. Some of the operational properties are given. It is seen that even for much simpler functions, the finite Fourier cosine and sine transforms are unwieldy and in many instances. The inverse is given in terms of series. Many of the partial differential equations with boundary value problems are solved using the finite transform.

16.20 Key Terminology :

Transform pair - modulation theorem - convolution integral - Parseval's theorem - Dirac Delta functions - Gaussian functions - Finite transforms - Operational Properties.

16.21 Self Assessment Questions :

1. Find the complex Fourier transform of $F(x)=e^{-a|x|}$ where $a>0$ and x belongs to $(-\infty,\infty)$.

2. Obtain the infinite Fourier transform of $F(x)=\left.\begin{matrix} e^{i\omega x} & 0<x<b \\ 0 & x>0 \end{matrix}\right\}$

3. Find the Fourier transform of the step function $F(x)=\left.\begin{matrix} \frac{\sqrt{2\pi}}{2c} & -c<x<c \\ & |x|>c \end{matrix}\right\}$ Interpret the result. When c is zero.

4. Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda}$

5. Find the sine transform of $\frac{e^{-ax}}{x}$

6. Find the cosine transform of $t^n e^{-at}$

7. Obtain the finite Fourier sine and cosine transforms of $F(x)=x^2$ on $(0,c)$.

8. If the finite Fourier sine transform of $F(x)$ is $f_s(n)=\frac{1-\cos n\pi}{n^2\pi^2}$ determine $F(x)$ is $0<x<\pi$

9. Solve the Laplace equation of $F(x,y)$ over the interval $(0,\pi)$ with the boundary conditions $F(0,y)=F(\pi,y)=0; F(x,0)=0$ and $F(x,\pi)=k$ (constant).

10. Find the finite Fourier sine and cosine transforms of $\frac{\partial^2 U(x,t)}{\partial x^2}$ for $0<x<c$ and $t>0$.

16.22 Reference Books :

1. R.V. Churchill : 'Operational Mathematics' McGraw Hill 1958
2. B.D. Gupta : 'Mathematical Physics' Vikas Publishing House, 1980
3. P.P. Gupta, R.P.S. Yadav and G.S. Malik : 'Mathematical Physics', Kedarnath, Ramnath, Meerut, 1980